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MATHEMATICAL EDUCATION

BY

B. BRANFORD

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A Study of Mathematical Education

DIAGRAM OF THE DEVELOPMENT OF MATHEMATICAL EXPERIENCE IN THE RACE AND IN THE INDIVIDUAL

Time Axis runs from top to bottom through centre of Chart. Scale from 600 B.C., onwards, is roughly one inch = 2,000 years.

INTERPRETATION (see also chapter XVI).

The two elements of mental activity in the development of knowledge (excluding the will and the feelings) are:—

(1) Thought-activity, centrally excited (physiologically), forming the conceptual element of knowledge: this is symbolized in the diagram by shaded space.

(2) Sense-activity, peripherally excited (physiologically), forming the perceptual element of knowledge; this is symbolized in the diagram by unshaded or white space.

The conceptual (shaded) element of experience, postulating its existence, becomes ultimately unrecognizable in the chart, on the rough scale of ratio adopted.

The diagram is to be considered both vertically and horizontally, with a view to the suggested parallelism between racial and individual developments, and in respect of (i) the kind, quality, and quantity of the experience or knowledge, and (ii) the external factors stimulating the development.

Only the external factors have been summarized in the diagram. There remains the internal factor, the aesthetic-scientific interest impelling the mind to a study and perfection of its own creations—the pursuit of science for its own beauty.

ADDITIONAL EXPLANATIONS.

We may roughly group the main Primary and Derivative Occupations thus:—

I. Primary Occupations.

Miner.	Shepherd.
Woodman.	Peasant.
Hunter.	Fisher.

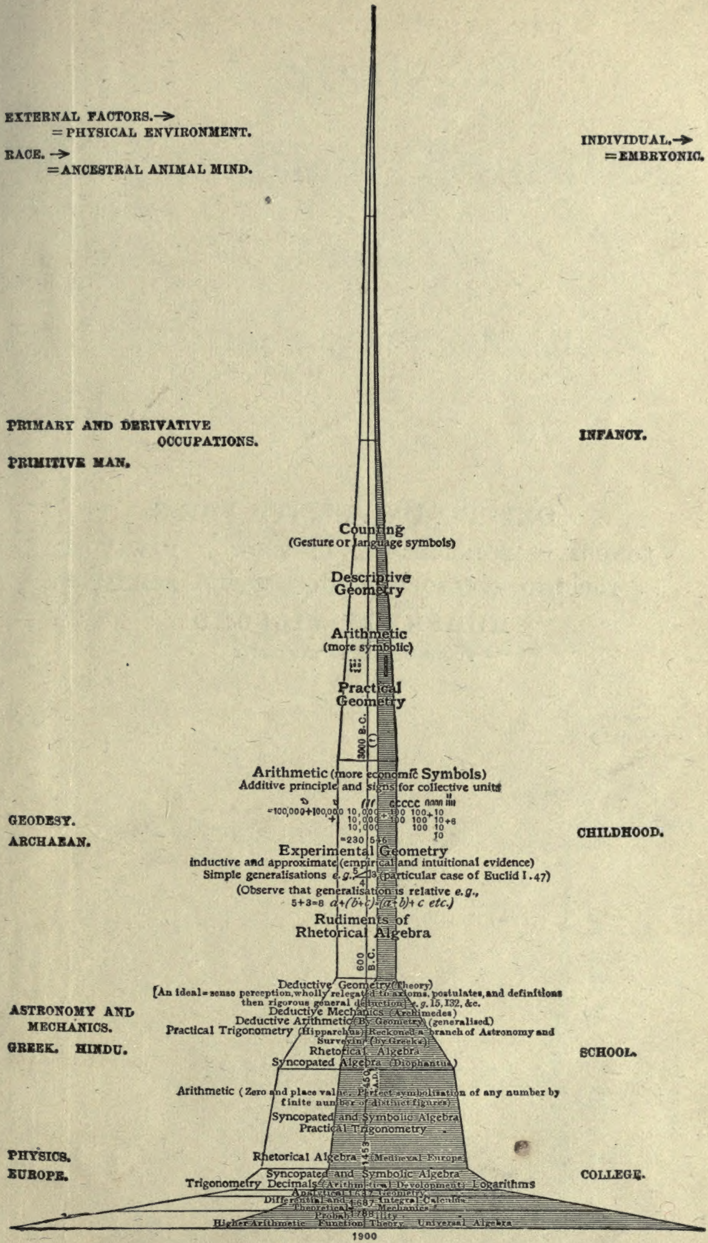
II. Derivative Occupations.

Trades.	*Professions.*
Arts and Crafts (male and female).	Scribe (Lawyer, Teacher, &c.).
Smiths and Wrights (Wheel- and Ship-wrights, &c.)	Musician and Poet.
Builder.	Warrior.
Engineer.	Architect and Engineer (e.g. Pyramid Builders).
Farmer.	Priest (Doctor, Teacher, Astrologer and Mathematician, &c.).
Trader and Merchant.	

Rulers and Administrators.

Under Geodesy would fall the Surveyor (of the Nile Districts) and Cartographer.

Under Archaean would fall Egyptian, Babylonian, Phoenician, &c. (Pythagoras was a Phoenician; possibly Thales too). Along with Hindu would come Arabian and mediaeval European mathematics.



EXTERNAL FACTORS.
=PHYSICAL ENVIRONMENT.
RACE.
=ANCESTRAL ANIMAL MIND.
INDIVIDUAL.
=EMBRYONIC.
PRIMARY AND DERIVATIVE OCCUPATIONS.
PRIMITIVE MAN.
INFANCY.
Counting
(Gesture or language symbols)
Descriptive Geometry
Arithmetic
(more symbolic)
Practical Geometry
3000 B.C.
Arithmetic (more economic Symbols)
Additive principle and signs for collective units
GEODESY.
ARCHAEAN.
CHILDHOOD.
Experimental Geometry
inductive and approximate (empirical and intuitional evidence)
Simple generalisations e.g. particular case of Euclid I. 47)
(Observe that generalisation is relative e.g.,
5+3=8 a+(b+c)=(a+b)+c etc.)
Rudiments of Rhetorical Algebra
600 B.C.
Deductive Geometry (Theory)
[An ideal=sense perception, wholly relegated to axioms, postulates, and definitions then rigorous general deduction] e.g. 15, 132, &c.
Deductive Mechanics (Archimedes)
Deductive Arithmetic (By Geometry) (generalised)
Practical Trigonometry (Hipparchus) Reckoned a branch of Astronomy and Surveying (by Greeks)
ASTRONOMY AND MECHANICS.
GREEK. HINDU.
SCHOOL.
Rhetorical Algebra
Syncopated Algebra (Diophantus)
450 A.D.
Arithmetic (Zero and place value. Perfect symbolisation of any number by finite number of distinct figures)
Syncopated and Symbolic Algebra
Practical Trigonometry
Rhetorical Algebra (Mediaeval Europe)
1453
PHYSICS.
EUROPE.
COLLEGE.
Syncopated and Symbolic Algebra
Trigonometry Decimals (Arithmetical Development) Logarithms
Analytical 1637 Geometry
Differential and 1687 Integral Calculus
Theoretical Mechanics
Probab 1788 ility
Higher Arithmetic Function Theory Universal Algebra
1900

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B.

A Study of Mathematical Education

including

The Teaching of Arithmetic

by

Benchara Branford

Author of 'Janus and Vesta, A Study of the World Crisis—and After'
and
'A New Chapter in the Science of Government'

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PREFACE TO ORIGINAL EDITION, 1908

This study is based upon twenty years' experience of school and college education ranging through all grades, including the very elements of counting and form in the kindergarten, the mathematics appropriate to the school, and the standard required of students for an Honours Degree in our universities.

Conclusions based on class-teaching and those arrived at by experiment and by a private study of the individual have served both to correct and to supplement each other.

The substance of the present volume has also formed the introductory part of a course of lectures on the teaching of mathematics, which, for many years, I have given in various institutions to classes of teachers from almost all types and grades of schools, elementary as well as secondary and tertiary.

I may perhaps be permitted here to acknowledge the benefit derived from the sympathetic criticism and co-operation of these teachers, in particular Mr. Hodgson, lately of Bingley Grammar School, and Mr. Walker, of Sunderland, the blind teacher of blind children. I should be ungrateful did I not also mention the stimulus received in my college days from the lectures and teaching of Professor Chrystal and the late Professor Tait, of Edinburgh University.

The principles here advocated are receiving daily verification in many schools, where the practical and theoretical aspects of mathematics are co-ordinated and developed: where simple descriptive geometry aids and is aided by clay-modelling and drawing: where theoretical geometry and practical geometrical drawing and mensuration illustrate and assist each other: where theoretical and experimental mechanics are associated with each other and with pure mathematics: where, in fine, all the branches of elementary mathematics, pure and applied, theoretical and experimental, are commingled at appropriate times, so that the mind sees and uses its mathematical conceptions and

processes as a beautiful, well-ordered, and powerful *whole*, instead of a thing of shreds and patches. Attention is now beginning to be paid in rapidly increasing measure to mathematics on its experimental and graphical side, and is exemplified by the use of drawing-boards, improved mathematical instruments, squared paper, and spherical blackboards. The provision of what may be called mathematical laboratories, well stocked with clay, cardboard, wire, wooden, metal, and other models and material, and apparatus for the investigation of form, mensuration and movement, is a refreshing sign to those who, like myself, have felt the general sterility and monotony, characteristic of perhaps most of the mathematical teaching of twenty and of even ten years ago, and still largely persisting at the present day.

It must, however, be admitted that the particular type of intellectual discipline obtainable from mathematical study on its formal, systematic, and logical side, is in considerable danger of becoming temporarily sacrificed during a too extreme swing of the pendulum of reform. Indeed, the time will doubtless soon be ripe for the serious consideration of the question whether a re-organization of mathematics as a whole—as contrasted with the geometrical synthesis of a Euclid—suited to the various grades of general education and technical training, is advisable, and if so, what is the most appropriate form.

In this volume I have attempted to indicate what appear to be the due functions of the two complementary aspects of the problem—the experimental and the demonstrative. On the basis of lengthy personal experience and experimental verification of the principles herein advocated, I venture to claim, not with undue confidence it is hoped, that the contents of this study are not mere phantasies of a pure theorist, but the sober resultant of long-continued action and thought.

The work has been composed in parts during the last ten or twelve years; it has been almost entirely rewritten several times, but doubtless contains much that is still very imperfectly worked out.

Further, the title of the work sufficiently indicates its limitations. Though bulkier than originally anticipated, the work makes no claim to be a systematic treatise on its subject, but should be regarded as merely a chapter

(though perhaps a lengthy one) on one out of many possible aspects of mathematics and mathematical education. Its ultimate object will be attained if it stimulates writers more competent than myself to undertake such a systematic treatise.

As for the more immediate object of the work, it is hoped that it may be of real service to teachers, students and bursars in our pupil-teachers' classes and training colleges, to teachers taking a post-graduate training, and to teachers of any branches of mathematics from elementary arithmetic upwards. Perhaps, too, in one or other of its aspects, teachers of allied subjects may find it suggestive.

A word of caution should be given to young readers respecting the unqualified form in which the heuristic, historical, and other principles have been stated. All educational principles are, in effect, ideals; and the degree in which they are realizable must depend upon actual circumstances—the enthusiasm, judgement, and skill of the teacher, the idiosyncrasies of his pupils, the practical limitations of his work. Experience alone can develop in a teacher true perspective as to the application of a principle. The realizable is ultimately the resultant of two forces—the strength of the ideal and the resistance of the actual.

Chapters XV–XVIII of this work summarize a course of unpublished lectures on the mathematical and physical exhibits in the Paris Exhibition of 1900, delivered for the École Internationale de l'Exposition. Chapters I–IV, XX, XXIII, with many and substantial changes, contain a reprint by the kind permission of the Editor of the *Journal of Education* (F. Storr, Esq.) of several articles written for that Journal.

I would earnestly plead for a greater appreciation of the value, to teachers as well as the general public interested in education, of historical study in scientific education—the history both of the particular science itself, which forms part of the curriculum, and of education in general.

It is a matter for regret that no adequate work on the history of mathematics is available in the English tongue: the contributions in this sphere of Peacock, Whewell, De Morgan, Gow, Allman, Chrystal, Ball, Cajori, and others, valuable as they are, have, I believe, exerted comparatively

little influence on the mass of teachers of mathematics. The great treatises are in foreign tongues, inaccessible to the majority of teachers; and even the best of these omit some important aspects. *Mathematical history has rarely been interpreted as an integral part of the historic movement of racial experience,* and appears *never* to have been so interpreted adequately. This last stage of interpretation may, perhaps, not be reached until women, too, have made their own special contribution to the development.

It is not, I think, sufficiently remembered that the history of mathematical science is part of the history of human education.

I am highly indebted to my friend, Mr. David Mair, and to the Clarendon Press itself, for valuable help and criticism.

BENCHARA BRANFORD.

24 February, 1908.

PREFACE TO NEW EDITION

IN this edition Part III is entirely new; but the changes in the original (Parts I and II), though numerous, are in general of subordinate details. A re-arrangement of the matter would have improved the order; but as modern progress has moved distinctly in the direction of the principles advocated in the original, especially those arising from historical, psycho-analytical (the function of sub-consciousness), occupational and relativity considerations, and further as teachers long accustomed to a book dislike any radical re-arrangement in its order, it was felt that the great expense of making such merely formal alterations would not in these days be justified. The author has been greatly encouraged by the world-wide welcome generously given to the work, including a pre-war German translation at the instance of the late Professor Klein of Göttingen, and a Russian translation in course of preparation by Professor Kulischer of Petrograd.

BENCHARA BRANFORD.

St. Patrick,
Whitstable, Kent,
18 April, 1921.

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CHAPTER I

MEASUREMENT AND SIMPLE SURVEYING

Part I

AN EXPERIMENT IN THE TEACHING OF ELEMENTARY GEOMETRY

Thales of Miletus (600 b.c.).

Thales, the founder of Greek geometry, discovered 'some things in a more abstract manner, and some in a more intuitional or sensible manner'.—(Proclus.)

1. I proceed at once to narrate, in as suggestive a way as I can, some details of an experiment I tried with a small class of children (average age ten) in the teaching of elementary geometry. Where my experience appears to me to have fully proved the value of some general principle, I have forthwith drawn attention to it, even at the risk of breaking the continuity of the details of my narrative. I believe that in general the statement of a general principle is most influential when it appears along with the particulars that gave rise to it and the detailed applications that subsequently tested and verified its truth.

2. *A Formal Treatment of Geometry breaks the Continuity of Child-life.*

Any one who has attempted to teach any system of *formal* geometry to young children, and thoughtfully pondered over the result, has probably been much struck by two facts—first, the surprisingly complex spacial ideas, and power of using these ideas for practical purposes, possessed by children; secondly, in complete contrast with the first, the fact that they are exceedingly slow, and often totally unable to assimilate the *formal* treatment and

elaboration of spacial ideas. We feel sure, as teachers and psychologists, that somewhere there has been a *break* in this attempted development of their geometrical knowledge, a discontinuity of treatment, a chasm across which the child-mind has failed to leap, a lack of links of association by which the new spacial life (if I may so speak) may be hooked on to the old. All these metaphors are useful for my purpose.

3. *Is School-life to neglect Life before School?*

It appears to me that one of the most inspiring signs of the times in education is the growing feeling—and the attempt to realize it—that, when he passes under the school door, a child shall not feel as if entering into an unsympathetic, foreign world, where all is mysterious and artificial, out of which he passes periodically, with dazed ideas, only too glad to find himself once again amidst the familiar and the intelligible. We would have him, on the contrary, bring with him his outside ideas of the real world into our schoolroom, there to be explained and developed, thus forming a firm basis on which to help him build the superstructure which is to increase his understanding of that world outside the schoolroom—*the* world, *par excellence.*

4. *What Space-knowledge has a Child before entering School?*

What is the particular application of this great educational principle in our present task? It is simply that we must try to discover and appraise at its true value the kind and amount of 'geometrical' knowledge the child has already in his possession, and the way in which he has gained this mastery; for, in a very real sense, a 'mastery' it is.

Listen to the talk of even a very young child, say of five or six (long before he enters school for geometry). What kind of words and phrases does he use? Top, side, bottom, above, below, inside, outside, upon, here, there, shorter, taller, big, little, far, long, broad, thick, thin, solid, point, line, round, half-as-big, &c. These are taken at random; many, many others might be added to the list. I would ask for careful reflection upon these words and the manifold intelligent uses a young child makes of them. We very soon begin to see the surprisingly complex stock of geo-

metrical ideas—space-knowledge—the child has gained. Here, surely, our aim should be to help the child, mainly in his own way, by observation, simple reflection upon concretely presented phenomena demanding explanation, by tentative processes (error alternating with success—as in life), &c., to continue this exploration of the physical world with increasingly accurate observations and more definite ideas ; above all, we have at once the duty and the pleasure of opening to him wider the world of quantitative measurements by means of number.

5. *The Welding of Arithmetic and Geometry in Intelligent Measurement by Units.*

'Arithmetic and Geometry', said the great Kepler, 'afford mutual aid to each other, and cannot be separated.' This was the spirit of the ancients. 'Jack is bigger than Harry,' a child will remark : here we have vague quantity. 'Father is much bigger than Harry' : here is a successful attempt at greater precision. When a child has grasped the idea of a unit of measurement, and uses intelligently such phrases as 'Father is just nine inches taller than Harry, as nearly as I can measure', he has advanced a long way into the heritage of knowledge bequeathed to him by his ancestors. In place of vague 'how-muchness' or 'whereness', we gradually help him to substitute numerical statements, so that, after the discovery of the simpler numbers, answering 'how many ?' (one, two, three, . . .), in the nursery, the child at six years of age, say, passes rapidly into the use of number as the measure of spacial magnitude. Contrast 'Where is the stamp ?'—'It is on the table, near that corner,' with the stage wherein could be grasped the reply, 'the middle of the stamp is five inches from one edge of the table and seven inches from the other, in a corner of the table indicated.' In the first we have vague geometry of position, in the last definite numerical measurement of geometrical magnitudes. Observe, incidentally, that 'geometry of position' (*analysis situs*) is a branch still awaiting due development (see Table, facing p. 158). To emphasize more cogently this early and continuous welding of geometrical with arithmetical conceptions, contrast the sentences : 'May I have some nuts ?' 'How many nuts may I have ?'—'Five.' 'I have half milk

and half water to drink at tea.' 'Half an orange.' 'My foot is eight and three-quarter inches long.' There is here a gradual passage from vague 'how many' to precise 'how many', from mere plural to definite number, then from number as 'how many' to number with vague units (cup or orange), as 'how much', finally up to the use of number as a measurement of geometrical magnitude in terms of units, themselves precisely measured (inches). The advisability of rapidly reaching in education the use of number, not merely as multitude, but as magnitude—i.e. not merely as 'how many', but as 'how much'—has been forcibly presented by the American writers McLelland and Dewey—an admirable combination of teacher and psychologist—in their much-to-be-recommended little work on *The Psychology of Number* (Appleton & Co.).[1] This intimate relationship between spacial ideas and number ideas, early in child life, is a fact not yet sufficiently utilized in school teaching. This must not of course be interpreted as advocating any neglect of the special development and the proper function of number purely as the 'how many'.

6. *A Rough Classification of a Child's Space-Knowledge.*

Even a very rough classification of the fundamental geometrical ideas of the young child may be useful and suggestive, and possibly pave the way for more thorough research. It surely behoves us, as teachers, to see to it that we carefully develop to fuller ripeness all conceptions already acquired by the child in his attempt to understand more and more thoroughly the world around him. To neglect a single one is undoubtedly a grave error, for all the conceptions already acquired are obviously those necessary and fundamental to the further interpretation of reality.

I. *Position:* (1) vague, e.g. outside, inside, above, below, to the right, to the left, near, beyond, &c.; (2) less vague, e.g. near the corner, just beyond, &c.

[1] To avoid possible misinterpretation, I add that, valuable as this work appears to be, the views of the writers with respect to the 'origin' of number—viz. 'in the adjustment of activity', a very vague phrase, for the authors' use of which I refer to the book itself—are, in my opinion, *far* from representing the whole facts of the case. Such attempts at the discovery of the 'origin' of concepts, in the present state of psychological science, are mostly nugatory.

II. *Motion*—Distance and time: (1) vague, e.g. quick, slow (of movements); (2) less vague, e.g. very quickly, far more quickly than . . .

III. *Geometrical Quantity*—Measurement: (1) vague, e.g. taller, big, little, long, equal; (2) undefined units, e.g. half an orange; (3) defined units, e.g. five inches . . .

IV. *Geometrical Quality*—Description, e.g. line, surface, solid, edge, sharp, blunt, circles, balls.

7. *The Child's Space-Knowledge is mainly used for Description or Identification.*

Generally speaking, one may fairly describe all this extensive vocabulary of ideas belonging to the child as descriptive, seeing that the purpose is seldom accurate measurement, but merely the desire of describing or identifying. Observe the vagueness from a geometrical standpoint evinced in such common sentences as these, 'Come out and run round the haystack': here we have respectively the geometry of motion, position, motion again, material object regarded as a perimeter. Again, 'A thick piece of bread with plenty of jam spread on the top': here we have six references to geometry. In such sentences we note the wonderful complexity of geometrical ideas attainable by mere infants.

8. *The Kindergarten.*

(1) *Simple Description.*—Gradually this descriptive knowledge is rendered more precise, and new ideas are added by a course of kindergarten training, when the child is asked and helped to describe the geometrical properties of simple objects, e.g. cubes, squares, boxes, balls, cylinders, &c. Herein the ideas corresponding to the words (corners) points, lines (edges), surfaces (faces), solids, &c., grow in wealth and precision.

(2) *Paper Folding.*—Then come paper-folding problems, training for eye, mind, and hand, where neatness and accuracy get developed, while simultaneously come notions of angles, parallel lines, right angles, blunt angles, sharp angles, perpendiculars, polygons, &c. There is a danger of over-elaborating this subject; it is a necessary preliminary—or let me rather say advisable—to higher development by the use of measured units, but may, if too long continued, easily pass into mere mechanical fooling wherein the mind ceases

to be active and power ceases to be developed. If the course is begun at a reasonably right age, a few months with a lesson or two of half an hour a week should suffice to yield all the educational value that is likely to be got from it.[1]

I proceed to describe, with the help of carefully written memoranda, made at the end of each lesson, an experimental course of teaching in elementary geometry, such as I spoke of, which may be of suggestive interest to many teachers.

9. *First Lesson* (*about thirty minutes*).

Sheets of paper distributed to each child; all asked to make a square. (The children had been through a simple course of paper folding such as I have described, and all made a square neatly.)

TEACHER (addressing a particular child; in future I omit remarks of this kind): 'What do you mean by the word "square"?'

CHILD: 'A square is a figure with four equal sides and with four right angles.' (This definition was not memorized by rote, but reached by actual observation of squares in paper-folding exercises.)

TEACHER (taking up one of the squares and bending it): 'Is this a square?'

CHILD: 'No.'

TEACHER: 'Why not?'

CHILD: 'Because it is not flat.'

We thereupon agreed to insert the word 'flat' in the definition of a square.

TEACHER: 'How can we tell when a figure is flat?'

Various answers to this, such as (1) 'When it is not like a ball,' (2) 'When it is smooth all over and not bumpy,' (3) 'When you cannot measure the height of it,' (4) 'By laying something flat on it,' (5) 'When it looks flat,' (6) 'When you cannot measure its thickness.'

Note here that (1), (4), and (5) have more elements of truth

[1] I here refer merely to what is commonly understood in kindergartens as 'Paper Folding'. The valuable exercises in the higher species of 'Paper Folding', such as are issued under the title *Geometrical Exercises in Paper Folding*, by Sundarù Row (Madras: Addison & Co.; London: Simpkin, Marshall & Co.), belong to a different grade of school life, although they may be used occasionally with advantage *throughout* the geometrical education.

in them than the rest; in (2), (3), (6) we have a confusion between the general idea of a surface and the particular kind of surface known as a 'flat' or 'plane' surface. This was more distinctly brought out by further questioning, when the child who gave answer (6) also added that 'a figure is flat when it has only breadth and length'. In these answers several paths are suggested as worth following out. The one that led ultimately to a satisfactory answer, indirectly dispelling the confusion and vagueness evinced in the minds of the children, was the clue given by (2), (3), and (6). The aim now was to help the children to find out, by appeal to objects around them, the distinction between 'surface' and 'flat surface'. In doing this they will be stimulated to distinguish surfaces from solids, lines, and points, and, again, flat surfaces from curved surfaces. We therefore proceeded to classify—for geometrical purposes (note the abstraction here from colour, hardness, weight, and other properties of a body, with attention only to form or shape; a rough list of the qualities neglected in geometry is, of course, to be got out of the children themselves)—we therefore proceeded to classify various objects in the room. Needless to say, this proved a very interesting task, and was entered into with spirit and success. The names of objects as observed and named (geometrically) may be now placed on the blackboard; the children can then proceed to rearrange them neatly into classes—a simple, yet truly scientific procedure, remark—with as little help as possible from the teacher, but with as much *relevant* criticism as possible of each other. Objects named: Floor, ball, top of desk, edge of desk, corner of desk, outside of ball, face, ear, pencil, mantelpiece, &c.

10. *Geometrical Classification of Common Objects, made by the Children.*

I. Points or corners: e.g. pencil-point, desk-corner, &c.

II. Lines or edges: (1) Straight lines or edges; e.g. desk-edge, lines between top of ceiling and wall, &c. (2) Curved lines or edges; e.g. edge of corner of mantelpiece, eyebrows or lips, ears, &c., chalk-line on ball.

III. Surfaces: (1) Flat or plane surfaces; e.g. desk, blackboard, floor, &c. (2) Round or curved surfaces; e.g. surface of ball, face, &c.

IV. Solids: e.g. body, ball, &c.

It was then seen that points (or corners) bounded or separated lines; lines (or edges) bounded or separated surfaces; surfaces (or faces) bounded or separated solids.

Problems: (1) Name some solids that have no edges. (2) How many edges has a box, a table with four straight legs? (3) Describe, or define, a 'square' more accurately (four straight sides, &c., flat).

Before proceeding further with these lessons, I think a few general suggestions may be acceptable. My aim throughout is to urge the teacher to a criticism of his principles and methods, to suggest possibilities, and to question received traditions.

11. *Children themselves can Help us to Determine an Appropriate Order of Developing Geometry.*

It will doubtless be observed, in the foregoing little exposition, that the order of the development of the subject is partly, and even mainly, indirectly decided by the answers of the children themselves. How far a teacher may safely trust to such accidental suggestions is a question often very difficult to answer. It is clear that, in a large class, where a variety of replies will occur, a considerable choice, at all events, is offered of pursuing a particular line of thought; yet not one of these replies may coincide in direction with the plan predetermined by the teacher himself. The teacher's common sense will here be exercised in harmonizing due continuity of development with the order that may be suggested, on the spur of the moment, by the chance replies of the children—chance replies, yet of supreme value, because spontaneous.

As practical teachers, we know that the main road must be determined beforehand, while the objects investigated by the roadside, on the march, should be those observed by the children themselves. But though practical exigencies make this demand, we may be sure that the line of development of the main road (or main roads, for several are needed for the various conditions and grades of education) can only be wisely determined when based upon numerous experiments designed to discover the order most effective and natural for the pupil in his various stages of growth. Doubtless, here as elsewhere, when old authorities are overthrown,

there will be a period of confusion and anarchy until, in the struggle for survival, one or more systematic courses vanquish the rest and dominate mathematical education for a considerable period.

It has repeatedly happened to myself to have the entire lesson successfully changed in direction by the spontaneous question or reply of some bright and interested member of the class.

12. *Definitions are the Working Hypotheses of the Child; they Develop Gradually with the Growth of his Knowledge.*

To me it appears a radically vicious method, certainly in geometry, if not in other subjects, to supply a child with ready-made definitions, to be subsequently memorized after being more or less carefully explained. To do this is surely to throw away deliberately one of the most valuable agents of intellectual discipline. The evolving of a workable definition by the child's own activity, stimulated by appropriate questions, is both interesting and highly educational. Let us try to discover the kind of conception already existing in the child-mind—vague and crude it generally is, of course, otherwise what need for education?—let us note carefully its defects, and then help the child himself to re-fashion the conception more in harmony with the truth. This newer and correcter conception, sprung from the old, will itself subsequently be replaced by a truer, but it has thereby played its essentially useful function as a link whereby the vague becomes slowly transformed into the more accurate and true. Only thus can we make sure that the child assimilates knowledge and is really prepared for the digestion of more complex mental food. Contrast this procedure with that of forcibly thrusting into the mind a full-born definition of which the child neither perceives the need nor understands the beauty and the truth.

13. *Definitions are Never Perfected; the Crudest Descriptions serve in the Origins of Science, both in the Child and the Race.*

We may carefully and successfully avoid this grave error of supplying definitions ready made, and yet fall into the opposite extreme. This is an equally grave fallacy in method characteristic of the young enthusiast—the fallacy

of aiming at perfection, thoroughness, and complete mastery at the time. It is generally a more or less spontaneous and accidental combination of ideas that clears up an obscurity most effectually in a child's mind: this natural process cannot be forced. Hence, of set purpose, I say: 'Be satisfied with a provisional working definition.' Indeed, it requires but a moderate acquaintance with the philosophy of language to be aware of the great truth that such things as perfect definitions do not exist—not even in geometry. Once we have watched the gradual and slow procedure by which a child reaches a consciousness of the conventional (strictly approximate) meaning of words in his mother tongue, we have, as teachers, received a valuable object-lesson. At school we can and ought to expedite this process, but it is in vain we strive to alter its nature, which I take to be the gradual emergence of relatively greater clearness and accuracy, through the agency of numberless trials and experiences of a great variety of contexts.

14. *All Conceptions are Subject to the Law of Development.*

Every term that embodies a conception is subject to the fundamental law of growth or development, whereby, in friction with its fellows, its significance is ever gradually changing. Equally true is this of the language of the race and the language of the child. A dictionary that should attempt to give the almost infinitely numerous shades of meaning attachable in a single century to one single word would fill a bulky volume and not exhaust the meaning. It is commonly thought (and a misinterpreted saying of Kant's gives apparent support—the saying that mathematics begins with definitions; all other sciences end with definitions) that geometrical definitions are perfect. But we have only to reflect upon the ceaseless controversies that have agitated, and still agitate, mathematicians and philosophers with respect to that fundamental geometrical conception—the straight line—to be once for all convinced that geometry, too, shares the imperfections of her sister sciences, in this respect, though of course, in a minor degree.[1]

[1] See the Author's *Janus and Vesta,* Chap. XI (Chatto and Windus); also A. N. Whitehead, *The Concept of Nature* (Cambridge University Press).

15. *Geometry is No Exception to this Law.*

Indeed, the whole history of mathematics is one long-continued development of the implications of terms, not merely as respects the discovery of new truths, but in the reaction of these creations of thought upon the original meanings of mathematical concepts themselves. The upshot of all this is that, as teachers, it seems that we 'put the cart before the horse' when we hand to children, in subtly logical form, statements of definitions and axioms, &c., reached by Greek thought only after centuries of effort.

16. *The Lesson taught us by History.*

It appears that, before Plato and Aristotle turned their philosophical eyes upon geometry (neither was a professional mathematician: neither made original discoveries in the science: but each made valuable improvements in its logic), the preceding geometers had 'used axioms without giving them explicit expression, and the geometrical concepts, such as the point, line, surface, &c., *without assigning to them formal definitions*'. All had gone successfully and merrily onwards without attempt at analysis of foundations. Then came the members of the Platonic School, who appear to have created most of the definitions popularly ascribed to Euclid, and, probably, many of his axioms, too. Aristotle refers to Plato the statement of the axiom 'Equals subtracted from equals leave equals'. It is interesting to find, many centuries later (about 1100 A. D.), the famous astronomer-poet of Persia, Omar Khayyám, writing a work in explanation of difficult definitions in Euclid.

And so it ever is; rapid discovery in science invariably precedes criticisms on its logical foundations. How could it be otherwise? Philosophical terminology and strictly formal statement were but wind and chaff without a substantial basis of fact to work upon, obtained by naïve intuition and common-sense argument.

Let us regard our pupils as little pioneers in geometry, and treat their crude definitions and statements with the respect and gentleness of criticism which all thoughtful minds accord to primitive discoverers, in all sciences and in all times. We may, I believe, safely act upon the truth that, in mathematics, if the child himself is active in the creation of the thought from the stores of his own experience, then

the conception of the thing defined and the working definition grow towards perfection together in mutual interaction. 'I know of nothing more terrible', says Mach, 'than the poor creatures who have learned too much. Instead of that sound, powerful judgement which would probably have grown up if they had learned nothing, their thoughts creep timidly and hypnotically after words, principles, and formulae constantly by the same paths. What they have acquired is a spider's web, too weak to furnish sure supports, but complicated enough to produce confusion.' Equally stern is the verdict of Boole: 'Of the many forms of false culture, a premature converse with abstractions is, perhaps, the most likely to prove fatal to the growth of a masculine vigour of intellect.'

CHAPTER II

MEASUREMENT AND SIMPLE SURVEYING

Part II

First Lesson—*continued.*

1. *Lines.*

Before completing the first lesson, *we* developed to greater clearness the idea of a *line*. Asked to draw a 'line' on the blackboard, a child made this figure—

Fig. 1.

I draw a very thick line with the chalk—

Fig. 2.

The children at once began to discuss whether this was one line or two lines. We finally agreed that it might be *meant* for one line but really contained two lines (top and bottom); in fact that it was really a small surface, and that we could not draw *one* line on the board with the chalk without drawing *two*.

Teacher: 'Which line, or boundary, shall we take if we wish to measure accurately the size of this triangle?' Here

Fig. 3.

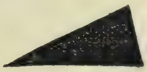

Fig. 4.

I chalked out a triangle on the board, with very *thick* sides (Fig. 3). At this point a child came up to the board, took

the chalk and completely *filled up* the space inside the triangle, thus, exclaiming triumphantly : 'Now we've drawn *one* line [meaning the exterior boundary] without drawing *two !*' I was decidedly pleased, of course, with this smart criticism ; but, before I could say a word in reply, another child (evidently anxious to save the teacher's reputation—children appear to be very tender on this score if one is sympathetic with them—partly, doubtless, eager to criticize the critic) interposed the remark : 'But you [referring to the other child] had to make two lines *at the start !*' This settled the question to their satisfaction ; and, after a little more discussion, the following statement was placed upon the board :—'In drawing the boundary of a figure, the bounding "line" is to be made thick enough to be seen, but not so thick as to take up space we can measure.'

2. *Straight Lines.*

'How shall we *describe* the difference between a straight line and a curved line ?'

Various answers, e.g. : 'A straight line always has the same direction and does not change.' 'A straight line is the shortest distance.' A question or two caused the addition to the last statement of the words 'between two places'.

The relative vagueness of the word *place* gives a capital opportunity of developing a more precise idea of the meaning of *point*, and of showing how convenient an idea and word it is for purposes of measurement and accurate description. The lesson was finished by putting a little problem :—'How many lines can be drawn through two *points* (placed on the blackboard) ? How many of these are *straight ?*'

This lesson occupied half an hour.

Second Lesson. (Briefer Notes.)

Henceforth, except where a specially interesting conversation took place, for brevity I simply state the lines along which we developed the subject. With rare exceptions, *a little appropriate stimulation educed the answer to, or solution of, the difficulty entirely from the children themselves.* I may remark here that, when (after two or three lessons) we began some simple surveying, some of the children, entirely of their own accord, so interested their fathers in the matter that (as I heard subsequently) the measurements

made at school were afterwards repeated by parent and child at home. The great interest taken by the children in this method of developing geometry was, in fact, very remarkable throughout: equally so were the originality and sharpness of the replies received.

3. *To each Teacher his own Way.*

In attempting to help others by an account of one's own failures and successes, one runs the risk of assuming an appearance of authority and even dogmatism. Especially is this likely to be the case where brevity is essential. If I am unlucky enough at times to appear in this objectionable guise, I sincerely assure the reader it is *merely* appearance. I have myself fallen into too many pitfalls in the path of the teacher to feel at all inclined to dogmatize; yet, as I believe that (with much effort!) I have climbed out again, I may hope to warn others of their existence, and, if some still struggle in the pits, I may perchance help them to get out.

No attempt is made to manufacture an infallible specific for perfecting mathematical education: the aim is much humbler. I hope that this account will lead others to test the value of my suggestions. I should like, too, to see others relating their experience. With a large amount of evidence thus collected from teachers of all ages and kinds of experience, there would be reasonable hope of deducing therefrom a body of principles, bearing upon the teaching of mathematics, which might really merit the title of *educational science.*

At present, however able and successful a teacher may be, it generally, alas! happens that his wisdom all dies with him; and, except for a comparatively small number of scholars whom he has educated, the fruits of his experience have no direct influence upon other teachers. And yet incalculable almost is the amount of experience that might be focussed on the testing of any stated methods and vaunted principles, to burn up their dross and refine their truth. Once we are agreed upon fundamental principles, then the detailed application of them is simply a question of the individuality of the particular teacher; freedom to the teacher thoroughly grounded in these principles to apply them according to his own particular experience—this surely

is an ideal of education. It is, therefore, very far from my opinion that the exact line of development herein to be set forth is always advisable in the treatment of geometry. It is simply offered as suggestive. In time I fervently hope that there will always exist some schools where the experienced teacher will construct his own syllabus. With this he will educate most efficiently; it will be the loved issue of his own labour. Moreover, only thus can there grow the seeds of those reform-movements which, from age to age, destroy the domination of a standard syllabus that has become antiquated—that has ceased to fulfil its original purpose efficiently. One thoroughly destroys only what one reconstructs.

The general monotony and frequent sterility of our educational methods point to deep-lying errors. I have striven to indicate clearly some, at least, of these; though I do not, of course, pretend to have discovered all—perhaps not even the most fundamental. I would modify the famous saying of Cousin, and urge that 'Criticism is the life of education'.

4. *Point and Straight Line: Position.*

'A *straight* line is the shortest distance between two *places*,' says a child. Teacher: 'How far is it from here [the school] to the church over *there*?' To be as accurate as we can, we find (1) it is necessary to state from what *part* of the school and to what part of the church we are to measure; (2) that it would be convenient to place a mark, say a chalk mark, on these two agreed-upon 'parts'; (3) that this mark must be made so small that it occupies no measurable space, and yet is big enough to be seen. Agreed then:—

'A point is really a small surface which is large enough to be seen but not large enough to be measured; a point is necessary as a mark of position, for it tells *where* to measure from.'

It was also clearly seen that, though both actual lines and points occupy space, we attend only, in measurement, to the lengths of lines, neglecting the breadths; while with points we neglect both length and breadth, and merely use them as marks of position.

I note here that, *previously* to this clearing up of ideas on

the conventional meanings of *point* and *line*, to the question: 'How many straight lines can be passed through two points?' was replied: 'Any number, if you make the points *big enough*!'—e.g.[1]

FIG. 5.

The convenience of taking the more restricted meanings of 'point' and 'line' above indicated was subsequently seen; this view, of course, led to the statement:—'It is convenient to use the terms 'straight line' and 'point' in such a way that (1) only one straight line can be got to pass through two given points, and (2) only one point is common to two given straight lines.

5. *The Drawing of Straight Lines—Units—Sighting.*

Two points marked on blackboard. Problem: 'Join them by a straight line.' Solution: Stretch a piece of string *tightly* between the two points. All agreed that the string must be tight, in order to get the *shortest* distance. Finally, with chalk, the child marks out the line indicated by the string.

Harder problem: 'Two pins are stuck into a long desk, one at each end (at the extremities of a diagonal, so that no natural line in the wood can be made use of), e.g.

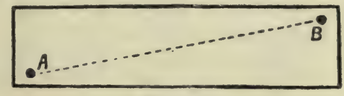

FIG. 6.

Required the magnitude of the distance between them.' (Materials allowed: foot-rule, divided into inches and fractions of an inch, and string.)

In simple problems like this the teacher will find that he has merely to look on and say nothing.

Solution: One child holds an end of the string at the pin *A*, while another walks to *B*, and stretches the string tightly between the two pins. The length used is then measured

[1] Such answers throw considerable light on the nature of the process by which children evolve the riper conception from the cruder.

by foot-rule as accurately as possible. Result: 14 ft. 6¾ ins.

Next Problem: 'If I want to measure the distance between here and the church, and have not a piece of string long enough, what am I to do?'

A similar problem is constructed for solution in the schoolroom itself, thus:—'Measure the distance between the two pins A and B in the desk; but no piece of string longer than a *yard* is to be used.' Here, for the first time, the children saw a serious difficulty. 'In what direction shall we start?' asked some. Others started off measuring from A by guess-work, but the brighter members criticized at once, saying: 'But you're going the wrong way; *that* is not a straight line!' (meaning the path $A\ C\ D$ . . .)

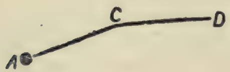

FIG. 7.

TEACHER: 'How do you know that AC is not a straight line that will pass through B?'

'We can *see* that it is not.'

TEACHER: 'Then it is evidently better for *one* of you to measure, and *one* to "see" that the measuring is straight for B.'

Teacher suggests sticking in a pin to mark the end of every measurement, instead of holding the finger there. The children now grasped the practical solution of such problems. Thus,

FIG. 8.

one child stands at the pin A, another at the pin B, each *looking* along the line joining the pin-heads of AB (the cheap steel pins with white-beaded heads are very convenient for this purpose). A third child then takes a number of pins and sticks them succesively into the desk at conveniently short intervals, according to the directions given by the other two, as at C, D, E, . . . , so that the pin-heads all appear to be in one straight line. This, of course, is simply what is known as 'sighting' in surveying. Every child, be it observed, was decidedly anxious to take a 'sight'. In a

large class several distances might be measured simultaneously, and the *observing instincts* of all be thus satisfied. They by no means all agreed as to the precise spot in which a pin should be placed, so that I had frequently to adjudicate between rival claims. The love of, and capacity for accuracy obviously varied much amongst them.

Here let me draw attention to the importance of instilling a few simple ideas on the nature of measurement (i.e. measurement of external magnitude, or what is known as *empirical* measurement). That such measurements are always *approximate* (an equivalent phrase for the child is 'not perfectly true or exact'); that different persons generally get slightly different results; that the *more patience and care given, the more accurate or truthful is the measurement*—these truths can be easily and forcibly brought home to children by observation of their own attempts. During subsequent kinds of measurement, where we had *two* methods of taking the same measurement, the results happened to differ. I said: 'Which are we to take as the more correct?' Two children at once said: 'Add them together and halve *it*.' Let it be noted that these children were beginning their geometrical studies *at least* a year before the customary age. This reply (though not really much more remarkable than many bright replies I received) I purposely mention as affording a piece of evidence distinctly encouraging to those of us who have faith in the power and originality of children when suitably guided and stimulated, and yet—*when repeatedly thrown on their own resources!* Here we have a case of children spontaneously hitting on the idea of an *average* in measurement—a simple enough idea, of course, but one which (if given at all) is usually thrust upon the child in an environment (the orthodox arithmetic book, in which a 'measurement' itself becomes as symbolical in character as the figures that symbolize it!) so artificial and abstract that he has little interest in the idea and still less appreciation of its value.

Let me here confess that, in the more youthful and sceptical times of my teaching life, such bright replies as these generally led me to make careful inquiries into the educational antecedents of the particular individuals who gave them—to discover if the solution was really original or simply the outcome of imitation. Further experience

pretty conclusively showed me the uselessness of this procedure, and continually helped to replace my scepticism by faith; *now* I receive with unquestioning gratitude these little gifts the gods send: may I venture to recommend this attitude to others?

Let me return to my class. By this method of 'sighting', the children succeeded in tracing a straight line between A and B. This they then easily measured by repeated use of the yard of string. Here turns up a little problem in *arithmetic*: 'A string, one yard in length, goes four times into a certain distance AB, and the rest measures 2 ft. 8 in. Find the whole distance AB.'

Result: 3 ft. × 4 + 2 ft. 8 in. = *14 ft. 8 in.* Thus may arithmetic and geometry mutually aid and develop each other. The day, I hope, is not far off when arithmetic will cease to be regarded only as an isolated 'subject' of instruction, and will take its proper place also as the handmaiden of geometrical and physical sciences. Doubtless we shall some day even speak of teaching *Mathematics* in place of arithmetic, in our primary schools.

6. *Flat Surfaces, again.*

'A *flat* or *plane* surface', said the children, 'is one on which straight lines can be drawn.'

I take a cylinder (such as those used in kindergartens) and ask if any one can draw a straight line on the surface of *that*.

'Yes.' One little pupil easily succeeds in chalking a straight line from top to bottom, with the help of string stretched tightly.

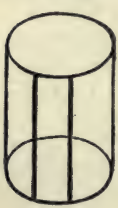

Fig. 9.

Teacher: 'Then a cylinder is *also* a surface on which straight lines can be drawn? Now point out some other surfaces in the room on which we can draw straight lines—but they are *not* to be *flat*.'

Lead-pencil, penholders, curved front edge of mantelpiece, cones, &c.—all these were indicated. To all such surfaces as a straight-edge or *ruler* could be fitted, we agreed to give the name of *ruled* surfaces.

'How then shall we distinguish between a 'ruled' surface which is also *flat* (or plane) and a 'ruled' surface which is *not* flat?'

CHILD : 'You can draw straight lines *all over* a flat surface but only in *some* directions on a cylinder.'

Between us (as usual) we therefore manufactured these statements :—Flat surfaces : test of flatness :—'A flat or plane surface is a particular and very common kind of *ruled* surface and is such that straight lines can be drawn on it in all directions through every point on it we like to take.'

Or we may put the matter thus :—'Through every two points we like to take on a flat surface a straight line can be drawn lying wholly on the surface.'

In contrast with plane surfaces are 'other "ruled" surfaces (e.g. cylinders and cones) through every point on which straight lines can also be drawn so as to lie wholly on the surface but only in *certain directions* ; in other directions these ruled surfaces are *curved*'. But a plane is not curved in any direction.

Our classification of surfaces now stood thus :—

Surfaces—I. Ruled [1] surfaces : (i) flat or plane ; (ii) partly curved (e.g. cylinders, cones, &c.) II. Not ruled, or, wholly curved surfaces, (e.g. balls, &c.).

[N.B.—An interesting and elementary discussion, relevant to this classification, is suggested by the form of the *human face*, e.g. the bridge of the nose in one child may be a 'ruled' surface, in another a 'curved' surface. Any attempt to still further describe differences of form apparent in objects, from the point of view here adopted, would soon lead to the introduction of *ideas* corresponding to synclastic surfaces (e.g. cheek of face, ball, &c.,) and anticlastic surfaces or cols, (e.g. *nez retroussés*, flesh between finger and thumb, &c.) : experiment only can decide whether these distinctions are too complex for young children to grasp. In the present instance I stopped short of this.

7. *How can we Measure the Breadth of a River?*

This last lesson, also, occupied about thirty minutes. I left them with the following problem to think about, if it interested them :—'I stand beside a tree on one side of a river which there is no means of crossing. On the opposite

[1] Incidentally note that the idea of a ruled surface is usually introduced only during an honours course in mathematics, at our Universities ! I have more to say on the importance of such ideas, hereafter.

bank stands a tree. How am I to find out the distance between these two trees ?'

At the beginning of next lesson the children made several ingenious suggestions—some decidedly naïve, which I was assured were their own : I quite believed them !—to get over the difficulty involved in this last problem. I mention one only : Tie a little stone to one end of a long piece of string, and throw it across the river : pull it back and measure the length of the string used. To this it was successfully objected by others that the river might be too broad. Other methods offered were equally futile. So they gave it up. As might be expected, none had hit upon the idea of using *angles* for the purpose. I forthwith tried to make them understand the usual method of solving this difficulty, but soon discovered that their previous very slight training in kindergarten descriptive methods had not developed the idea of an angle to a degree that enabled them to use it for *measurement*. Hence a few lessons were devoted to this necessary preliminary education. Incidentally, it will be noted, this procedure led insensibly onwards to the discovery (with which all were delighted) that the three angles of a triangle always amounted together to two right angles. But I anticipate ; of this more in the sequel.

8. *Angle : a Difficult Concept, essentially Quantitative.*

I note here that the promise of showing the way to measure the distance between our two trees, and to apply it, *when they understood the ideas it involved*, to two actual trees in the *playground*, proved a stimulant to curiosity and inventiveness of much power. The children felt they were working with a definite and practical aim : can a finer stimulant be found than this ? Why do adults have the monopoly of it? On examination, I found that their ideas about angles were vague in the extreme. It is indeed—this idea of an angle —a conception decidedly difficult (as most mathematical teachers soon discover) to form with any moderate measure of clearness. Though two or three of these children very rapidly attained an excellent conception of its meaning, with the rest I had distinctly felt difficulty. After some futile attempts with the duller wits —the futility was shown by their inability, still, to use the idea in measurement—I came to the conclusion that the

speediest method for bringing the development of their first crude idea to more clear and definite shape was simply to practise them in the making and measuring of various kinds of angles. Special care, it would appear, is desirable in the treatment of this concept, and plenty of practical measurements. Slowly the idea gains clearness: it is seen to be an essentially *quantitative* concept.

9. *Measure the Mind and Match its Complexion—by Observing the Child's own Efforts.*

Here, especially, do I believe that my criticisms against what I hold to be the abuse of definitions have sound warrant. Definitions of an angle, presented ready made, whether learned by rote or not, can much less fit the mind of the child than can ready-made clothes fit his body. 'Yet a nice object a child would look if he made his clothes himself,' perhaps some may be inclined to retort. Now, although it is quite possible that the attempt to make his own clothes might form an educational discipline excellent for the child, I do not venture to press further the analogy—at best, but slender—but merely ask: 'To clothe a child suitably, we *measure his body and match his complexion: shall we do less for his mind?*'

And what is the tape for the measure of his mind, or the means of judging its complexion? Is there any other than simply observing the deliberate attempts of that mind itself, in response to our queries, to use more correctly, and frame with clearer meaning, the words it already uses with uncertainty and vagueness? Or is it that children's bodies and complexions differ so decidedly, while their minds are practically similar? How long shall we regard every child as an average mental effigy on which to hang ill-fitting, ready-made educational clothes? Is Harry—stout, broad, breezy, coarse, blue-eyed, robust—of *mental* constitution so similar to that of Fred—small, thin, nervous, large-eyed, and delicate—that the same mental training will fit both, though each would look a scarecrow in the other's clothes? 'But', it is said, 'we must have fixed methods for all, adapted from the needs of the *average* boy.' That all boys possess certain fundamental qualities, is, of course, obvious; but who or what this *average* boy is I have never succeeded in discovering! Doubtless while the number of pupils in a class

continues to be so large—particularly in the elementary schools—education must imitate the manufacturer instead of the *artist*, and be satisfied with numberless copies of a few sizes of 'ready-made clothes'.

Here I abruptly leave the difficulty to the thoughts and experience of my readers. A day, I hope, will some time arrive when we cease to be forced to sacrifice either the extremes to the means, or the means to the extremes. The above remarks were made à propos of the general custom of supplying children with ready-made definitions in mathematical education.

10. *For Beginners great Precision in Definitions and Proofs is Injurious Pedantry.*

Some further explanation seems desirable to obviate misunderstanding of my position. I am well aware that the phrase the children gave me, 'A straight line is the *shortest* distance between two points,' is pleonastic. From a critical standpoint, one would, of course, omit the word 'shortest'; but to do so, without assigning sufficient reason, were clearly dogmatism: now in this case the reason is far beyond the capacity of beginners. Here, then, is an excellent instance where precision is mere pedantry.

Another example that forcibly illustrates my meaning is afforded by the 'amended' definition of a *square* which the children, under stimulus of criticism, manufactured: to wit, they were induced to add the descriptive word 'flat' to their previous definition, which ran thus: 'A square is a figure with four equal sides and four right angles.' Now, as a matter of fact, this, their first definition, just gives a minimum number of properties sufficient to distinguish the class of figures called 'square' from all other classes! For a figure with just these properties can be *proved* to be a *flat* (or plane) figure [1]. But the children did not know this; moreover, to establish its truth is quite beyond the capacity of beginners. Consequently, as the satisfactoriness of this statement, regarded as a logical definition, was clearly a

[1] I once received in an examination the definition: 'A square is a plane figure with all its sides equal and all its angles right angles.' After which followed a perfectly correct *proof* (depending of course on I. 32) that the number of sides must be *four*. It appeared, too, on inquiry, that this quaint definition originated from the fancy of the lad himself!

simple fluke, it appeared to me to be advisable to offer criticism, and get the children to see that, *from their point of view*, it needed a slight improvement. They therefore agreed to the amendment : ' A square is a *flat* figure,' &c. ; yet the addition of this word certainly caused the definition to be *redundant*, from the highest critical standpoint. From that point of view the definition had deteriorated ; from the children's standpoint the addition was a distinct improvement.

11. *Vagueness and Looseness in the use of Technical Terms is an equally Injurious Extreme.*

There is some danger [1] at the present moment of geometrical education running to the other extreme—vagueness and looseness in the use of terms. This would be almost as unfortunate as pedantic precision, though doubtless, as in all reforms, this extreme is inevitable before the reform reaches a sound and stable condition. Let us meantime simply remember that continual use reacts on the significance of words, and that a stimulating treatment of geometrical matter that is constantly testing correctness of hand in measurement, and demanding the clearing up of confusion in thought by appeal to the mind's own resources, will ultimately and unconsciously lead to a mastery of meaning.

Although at no one stage of the development may the *full* meaning of a term be grasped, this does not imply even the temporary tolerance of looseness and vagueness of meaning ; for, relatively to the extent of the child's knowledge at each particular stage of education, his use of every technical term may be clear and precise. Yet he will know much more about that term in due time, *afterwards*, and find generally that his first conception was crude, though sufficing for his needs at the time.

The following observation, made by myself and recorded at the time (of course, without the child's knowledge), is interesting in this connexion. It shows that precision and

[1] One must confess that there are now many signs that the reaction is advancing to this extreme in many schools. The need for a crystallization of the present reform-movement in a small number of systematic courses is daily becoming more apparent to the observer. At the same time there are many antiquated backwaters still untouched by the freshening breezes of reform.

clearness are not foreign to quite young children. The child was a little girl a few months over six years old—still untaught to read or write; in fact, still unschooled. The child's father is lying in a huge hammock, so large that a considerable portion of it, *at top and bottom*, is unoccupied.

CHILD: 'There 's a lot of hammock left for you yet, Daddy.'

FATHER: 'Yes; but I'm not *big* enough to fill *all* the hammock.'

CHILD: 'You mean you're not *long* enough, Daddy.' (With much stress on the 'long'.)

Note the *geometrical* ideas herein involved.

12. CONCLUSION.

The writer regrets that the great demands on his time of other and more important duties prevented him from continuing and completing these little articles descriptive of the experiment, whose beginnings have been therein described. It is now too late to finish the task, for, though notes of the actual lessons were taken at the time, the lapse of a few years in fully transcribing them is fatal to that freshness and accuracy of interpretation without which the description of the experiment is of comparatively little value.

Suffice it here to add that theory and practice went hand in hand throughout, so that the class was able finally to measure with tolerable accuracy the distance between two inaccessible objects by several methods, in each of which the fundamental geometrical truths (concerning the identity in size and shape of triangles, and simple metrical properties of similar figures) were not only mastered, and applied therefore, rationally, to the concrete, but actually *discovered* by the children themselves under the guidance of the teacher and the stimulus of knowing what interesting and useful practical applications could be made of them. Throughout the course (which occupied one term, with two lessons per week) all the children showed the keenest interest and some (I afterwards discovered) repeated the experiments entirely of their own accord in their own gardens and the fields round their homes—measuring with considerable patience and obvious joy the heights of trees, of their own houses, and so on.

CHAPTER III

FURTHER EXPERIMENTS IN THE TEACHING OF ELEMENTARY GEOMETRY.

1. *On the Importance of Ruled Surfaces.*

PERHAPS it may appear premature, and even ridiculous, to some, to introduce so early (as I did) the idea of a *ruled surface.* It is obvious that I do not share this objection. Why? For these reasons—which I trust will appeal to other teachers as forcibly as they do to myself. In the first place, I have found repeatedly that, unless planes are contrasted with other ruled surfaces, the notions called up by the word ‘ plane ’ are extremely crude, mainly formal and often even fallacious. Both as teacher and examiner have I experienced the truth of this. If inquiry be particularly directed to the fallacy, by some stimulating question, it will be too often found that even mature lads and girls, who have very correctly learned the formal definition of a plane and worked through a substantial course of plane geometry, are still under the fallacious impression (identical with that of young children) that a plane is the *only* surface on which straight lines can be drawn. Moreover, I have also observed, too often to regard it as a chance freak, a phenomenon of really astounding import in education were it not so fatally common: to wit, that these lads have become so stupefied by the severely formal training received that they show distinctly less capacity for discerning rapidly and clearly the precise points of difference—between, say, a cylinder and a plane—relevant to the inquiry than do young children many years junior and untried in geometry. The longer such fallacious impressions remain dormant, the harder it is to rectify them.

Then, again, the idea of a ruled surface is easy of appre-

hension to children. *Fiat experimentum !* Thirdly, consider the immense importance of ruled surfaces in actual life—industrial and other. Indeed the child is already familiar with such surfaces—are not many of his playthings bounded by them ? The commanding position held by the cone and cylinder appears to be due to the intimate relation they have to the plane. If I make no mistake, very simple geometrical considerations suffice to show that the cone and the cylinder are the only surfaces that are each of them at once ruled surfaces, developable surfaces, and surfaces of revolution.[1]

2. *A Blind Mathematician. The Fluidity of Figures.*

At the beginning of the eighteenth century, the famous Lucasian Chair of Mathematics (Newton's Chair) at Cambridge was held by one Nicholas Saunderson. This man, a successful teacher, and, in his time, a noted mathematician, was completely blind. He lost both eyeballs, before he was twelve months old, by small-pox. How did this man construct figures in teaching geometry ? In the memoir prefixed to his work on algebra (published 1740) we learn that he used a board pierced with a line of holes round each edge, into which were inserted pegs. Round these pegs were attached strings, stretching across the board, whose directions, by means of the movable pegs, he could easily alter at pleasure. By these means any straight-lined figure could be constructed.[2]

The great advantage such a device possesses over chalk-made figures lies in this : lines are so readily altered that the mind is thereby induced to consider a variety of cases ;

[1] Developable surfaces are those which, without *tearing* or *stretching*, can be rolled out into a plane. Cylinders and planes may both be regarded as particular cases of the cone. The sphere is obviously not a developable surface, though it is a surface of revolution. In industry, another important species of ruled surface is the common screw-surface (e.g. the under surface of a spiral staircase), but this is not developable into a plane. All these notions, with proper experiments, will some day be commonplaces in *elementary* education.

[2] We are not told how he constructed curvilinear figures, such as circles. The method above suggests an admirable way of constructing curves as *envelopes*—a simple idea, complementary to the construction of curves by points, which (as I have found on trial,) can be early and fruitfully introduced in the teaching of geometry.

the usual stereotyped form of figure disappears, and, with it, the mechanical attitude of mind that it engenders.

Further, *limiting* cases are thus suggested for discussion: e.g. in the theorem that the angle ACD = angle CAB

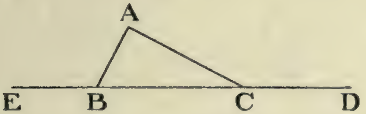

Fig. 10.

+ angle ABC, what special form does the theorem take when, AB and ED being kept fixed, the point C is moved along ED to an unlimited distance towards the right or left?[1]

I give these not as specially valuable examples, but simply by way of suggesting to the mind, whenever opportunity offers, the fertile idea of *continuity* in geometry by adducing visible evidence of the infinite variety of possible figures and of the marvellous way in which they can be made to *flow* continuously into one another. The awakening thus of the mind to this fluidity of figures assuredly tends to preserve its elasticity, and is highly conducive to an inventive attitude.

[1] A simple instrument for effectively suggesting this fundamental idea—the continuity of figures—is easily made by lightly fastening together at their extremities, with smooth pins, four unequal rods in the form of a quadrilateral, as in Fig. 11. This, of course, can be continuously deformed into various shapes. Actual experiment will first suggest which of these (if any) are concyclic, while theory will afterwards rigidly verify the suggestion. The fact that *three* rods, thus pinned together, form a figure (a triangle) which is *not* deformable is, of course, a fact equivalent to Euclid I, 8. Still more simple and instructive is it to have four stout pins which can be readily fixed in any required position on the blackboard. Round these (with a non-slipping knot on each pin) pass a piece of string, and you have a piece of mechanism so simple that each lad can make one and experiment with it himself. Such devices I have found very effective in treating the simple properties of a circle.

Fig 11.

It is one of the key-notes of modern mathematics—this continuity of treatment.

3. *Angles: Simple Instruments for Making and Measuring them.*

A convenient instrument for developing the idea of angles is a simple two-foot rule, treble-jointed, and thus divided into four parts of six inches each. Better still (as the thickness and breadth of the rules generally sold are not conducive to accuracy in measuring angles: one cannot get good coincidence with the *vertex* of the angle), the teacher should make for himself four thin narrow strips of wood, of equal length, pinned together pretty firmly at the extremities, (see Fig. 12) and such that, any limb being fixed, an adjacent limb can be readily turned through a whole revolution of four right angles. In Fig. 13, Aa coincides with AV, and Bb

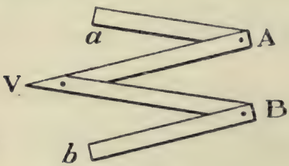

FIG. 12.

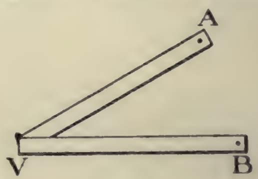

FIG. 13.

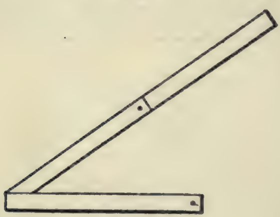

FIG. 14.

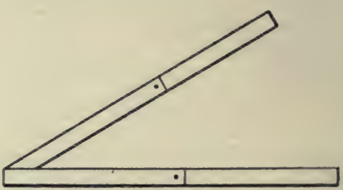

FIG. 15.

with BV. Fig. 14 is derived from Fig. 13 by opening out Aa into line with VA. Fig. 15 again comes from Fig. 14 by also opening out Bb into line with VB. The double limbs are very effective in helping to create the conception of an angle as a magnitude that is quite independent of the length of its bounding arms.

Another equally useful and still simpler instrument for developing the idea of an angle to greater clearness is an ordinary rectangular sheet of paper, which, by proper *creasing* as required, makes a good protractor or angle measurer—this may be used along with the other instrument. With it the child can be taught to make a good sharp vertex to the angle; obviously it is also very cheap and always at hand; so that each child may be supplied with plenty of these excessively simple paper protractors. The other instrument is more adapted to the teacher's use for purposes of illustration: it is not an instrument for exact measurement. What now is the *general* principle, relevant to the science in hand, that is to dominate our aim as teachers? I take it to be this: that skill and accuracy in measurement and clear understanding of the underlying rational process shall mutually develop each other in the child, by his own attempt to solve little practical problems presented to eye and hand, and simultaneously, under the stimulus of questioning, to gradually construct with explicit language the abstract theory or counterpart that rationalizes the manual skill; each aspect of power a deformity if it lacks the other—concrete skill and abstract insight.

Further, the *particular* inquiry we have in hand is to get the child to grasp the various aspects involved in the conception of an angle.

4. *Different Aspects of an Angle.*

First Experiment.—I place the hinged protractor against the blackboard; it is completely closed up and presents the appearance of a broad line. Then, keeping the two extremities a and b coincident with V, the vertex, with one hand I slowly revolve VA about V (in the plane of the board), meanwhile holding VB firmly fixed against the board with the other hand.

'What have I made here?'[1]—'An angle.' I ask a child to come and draw two chalk-lines on the board to mark the positions of the arms, viz. VA and VB. I now take away the protractor from the board, leaving the chalked angle.

[1] I remind the reader of the fact that the conception of an angle was not entirely new to these children, as they had gone through a very brief course of simple paper-folding antecedently. My aim was not to introduce a new notion, but to clarify and develop an old one.

'What have you made on the board ?'—'Another angle.' 'Has it any relation to the one I hold in my hand ?'—'Yes, it is a copy of it.' By this, it appeared, they meant that the two were equal in size. The reader will observe that these are two distinct ways of making angles, and indicate quite different aspects, one marking the amount of turning, the other marking differences of direction ; we have to develop both aspects, for they are equally useful.

The first (the turning or revolutional) aspect appeals to the idea of motion, and, therefore, involves the idea of *time ;* it might be called the 'kinematical' view of an angle. The second (the directional) aspect is independent of the idea of time, and is, therefore, more purely geometrical ; it might be called the 'statical' view of an angle.

Further Experiments.—I now *close up* the protractor in my hand, and ask the children to come and open it out to the *same extent as before.* The first who tries opens it out *at a guess !* 'But are you quite sure it is turned through the same angle as before ?'—'No !' 'How can you make sure ?'—'——' Another child comes to the rescue. '*By putting it against the blackboard angle.*' This the child does, and finds her angle is a little too small. 'Here, then, on the board I have a picture by which I can get back the angle I made with the protractor, if it happens to get closed.' I go to the board, and, with chalk, produce the arms VA, VB, making them longer. 'Is the angle altered at all by doing this ?' Some say 'Yes !' emphatically. Others say 'No !' equally emphatically. On cross-examination no satisfactory or precisely stated reason could be given by the 'Yes !' partisans. But the 'No !' party suggested comparing the apparently new angle on the board with the protractor angle which I held. It was then seen that, although the arms of the chalk angle were lengthened, it still served the purpose of getting back our old angle. 'Then I have *not* altered the size of the angle by lengthening its arms ?' All but one agree heartily ; that one is dubious.

I again take the wooden protractor, and, opening out an angle, ask them to observe carefully while I proceed to also open out the other limbs Aa, Bb, meanwhile keeping VA and VB fixed ; so that, finally, the instrument appears as in Fig. 15. 'Has the angle altered when I did this ?' 'No,' from *all.* 'Then an angle is not altered in size, to whatever

distance I lengthen the arms ?'—'No.' 'Not even if I lengthen the arms to a yard ?' 'To a mile ?' 'Ever so far ?' (accustom the mind to confidence in dealing with large magnitudes). 'Need the arms be of equal length ?' (protractor shown as in Fig. 14). *Agreed, then, that the size of an angle does not depend upon the lengths of its arms.*

By simple illustrations got from roads and paths near the school, and familiar to the children, they will readily see the close connexion between the idea of an angle and the idea of *direction.* In fact, that an angle serves to measure the difference in direction between its two arms; also the amount of turning[1] required to make one arm coincide *in direction* with the other.

Thorough familiarity with the successful and intelligent use of a conception—that is the kind of knowledge we wish to develop. The deliberate attempt to commit to memory the exact words of definitions, &c.—initially presented on the authority of the teacher—even after careful discussion and statement of them, I, for one, unhesitatingly regard as educationally vicious in the case of young children.

[1] Here, to illustrate further my contention that *sufficient for the day is the precision thereof,* I take it to be over-subtlety to insert the condition, 'the turning to be *in one plane*', or its equivalent, 'the *least* amount of turning'. This condition all children will *implicitly* satisfy in actual measurement; it appears much too early to force upon them conscious recognition of it.

CHAPTER IV

FURTHER EXPERIMENTS IN THE TEACHING OF ELEMENTARY GEOMETRY.

1. *Passive Experience or Deliberate Experiment—which is it to be in Education?*

In education that comparatively passive experience which seldom gets utterance has long proved itself a slow horse to ride; let us then, admiring the gigantic strides of physical science—a much younger figure on the stage of civilization—try the mettle of *its* charger, deliberate experiment. The characteristic of passive experience is to steer for nowhither; we simply drift; true, the outcome is often skill, but it is apt to be purely personal, incommunicable. The first condition of receiving a definite answer, of reaching definite, communicable truth, is to put a definite question. This is the first step in experiment.

2. '*Prudens interrogatio dimidium veri.*'

How many teachers ask themselves: 'What precisely are my aims in teaching this particular subject? What is the educational worth of both aims and subject? How can they best be realized? What limitations are imposed upon the realization of the educational worth of this subject by my own particular method of handling it here and now?' I name only a few dominant questions: to answer any one of them, numberless others, of more particular trend, must first be put—and answered. Answered how? By deliberate experiment only.

3. *Above all others, Educationists must be Co-workers.*

In building up a science of education, the unaided work of the single teacher counts perhaps for less than in any other science. Here, above all, are needed the criticism and construction that come of co-operation. We need a society for

advancing the science of education, which shall periodically publish original monographs, written by actual teachers on particular experiments. There is a vast number of teachers whose lives would become interesting and ennobled by such work, teachers whose talents too often for ever lie hid under a bushel.

4. *Emotional Impulse and Scientific Method must co-operate in Education.*

I speak from the broadest standpoint I can find when I express my conviction that the two fundamental weaknesses in our educational work are the lack of the higher emotional impulses and the lack of scientific method. Some may think these two things inherently antagonistic; but this I hold to be a shallow view and so far from the truth that only, I believe, by their *union* can education become a really potent factor in the advancement of man. Without scientific guidance, the emotional impulse, however high and deep, is blind: without an accompaniment of such higher emotions, method, however scientific, is a deadening machine. What share does education really get of those energetic impulses that spring from the teacher's lively interest in his work, from conviction of its efficacy in developing power and character in the pupil, from independence in handling his task to the best of his ability, from public esteem of his services—from endless other sources of the higher emotions? Were I to make against our educational system the deliberate charge that we neglect the *higher emotions, both of teacher and of pupil, as the impulses to education whereby its greatest energy is communicated and its life-long continuity is guaranteed to both*—to what degree could such a charge be truthfully rebutted?

5. *Does or does not a Process of Real Education simultaneously produce obvious Increase of Power?*

Many educationists, looking to the apparently insignificant direct effects produced in the character and power of the pupil while under a process of school education, urge upon the teacher the need of faith in the ultimate outcome of his work. Faith, we must have; but, under such circumstances, is it not wholly misplaced? Is not the wish here father to the thought? Such faith, without good grounds, can never work wonders: indeed, I believe that among

teachers there is little of it really operative. Rather is there despair, which obvious facts appear to warrant. Indeed, if we try to analyse the grounds for this faith recommended to us, our confidence in it gets rudely shaken. What is the underlying idea ?—that after many years, when the character has reached comparative maturity, the good effects of the education will put in an appearance. (Here I ignore as irrelevant those aspects of school education not directly concerned with moral character and personal capacity.) But is such a phenomenon credible ? What evidence is there for it ? Does any effect thus suddenly exhibit itself ? The supposed fact is contrary to the well grounded principle of continuity. 'Natura saltum non facit.' For some purposes we compare education with the seed of a tree that takes long to mature ; but would it not be foolish to keep planting seeds where never a young shoot appears, in the hope, forsooth, of waking up some fine morning to suddenly find a full-grown tree on the spot ? No, let us teachers have the pleasure, the interest, the impulse, the confidence which can come only from actually watching the increasing growth of the young shoot. In plain language and common sense, let me see that my educational efforts really and continuously do develop the inventive power of the pupil, do excite his interest, do minister to his innate desire for intellectual mastery, do increase his initiative and independence —all of them to a degree that is something very different from the present negligible quantity. If you merely preach truth to a young child, taking little heed the while that he practise it, is it reasonable to expect that child to become at last a truthful man ? That inborn quality and power is prone to decay and die which education does not deliberately stimulate. With few exceptions, I believe, our strongest, finest impulses to effort exist *outside* the schoolroom ! What a strange phenomenon in education is this !

6. *Dried Nuts and Living Trees.*

The beginnings of all things are admittedly the most important. Therefore I interpolate here a short description of other experiments on angles with a fresh batch of youngsters. Here, though the principle that dominated the teaching was the same as in other experiments—the development of character and knowledge in the child through his

efforts to rationalize his geometrical experience—yet the details, the arrangement of the material, the instruments used, were more or less different. I insert the chapter by way of suggesting that the teacher should never stereotype a course. The needs and individualities of new pupils will ever unconsciously suggest to the observant teacher new arrangements—a new syllabus. The interest to the teacher in the gradual development of ever new aspects of presentation affords strong presumption that he will interest his class—the first condition of educational efficiency. We are told that the genuine teacher is also an original investigator: true; and is not this the very sphere in which his originality is ever exercised? Those high emotions that accompany the creation of a thought, however humble it be, are as inspiriting and nutritive to the teacher as to the taught—and as necessary. Let us out into the fields and gather for ourselves the fresh fruit from the living trees of knowledge! Not geometry, still less Euclid, are we to teach; our pleasant task is no less than teaching the child the interpretation of the world as geometrical.

7. *Making and Transferring Angles.*

Problem—(Sheets of paper distributed amongst class): 'With the help of your sheet of paper make an angle equal to this angle that I have drawn on the board.'

After some difficulty the class succeeded in this by properly *creasing* the sheet of paper. This formed a little exercise in perseverence and accuracy: only by 'trial and error' was it to be done. Let the teacher then place a *transparent* piece of paper quite over the angle. It is easy now to trace two lines on the paper just covering the two arms of the angle on the board. Here, then, we have our angle on the transparent sheet. But how can we *transfer* it to another assigned part of the board *without creasing?* This is not so easy. At length, by simple questions, you lead the children to discover what is to them a remarkable fact. The paper, let us suppose, is held closely against the blackboard in a new position. To transfer the angle we have *merely to prick three points in the paper*. Whereabouts? One anywhere on each arm, and one over the vertex or corner of the angle. Three marks are thus left on the blackboard. The child removes the paper, and has now merely

to join the three points *suitably* by a pair of intersecting straight lines. The angle has been transferred. Simple as these ideas and movements are, they form an exercise sufficient to tax the skill of most children nine or ten years old.

8. *The Development of a New Geometrical Truth and the Literary Exercise in Self-expression accompanying it.*

After a few simple experiments of this nature, it is perceived that, to transfer angles by this principle without the use of creasing, one can dispense altogether with the *transparency* of the paper. Simply choose a piece of clean paper of such size that it can be readily placed as indicated in the figure. It may be of any shape, the size alone is important.

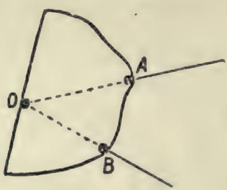

FIG. 16.

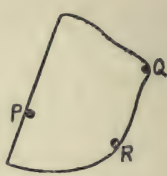

FIG. 17.

Bring any marked point of its boundary into coincidence with O, the corner of the angle to be measured and transferred. Now mark the two points (A, B) in which the arms of the blackboard chalked angle cut the boundary of the paper. (Fig. 16.) Our simple protractor on being removed and transferred to a new position appears as in Fig. 17. Mark corresponding points P, Q, R on the blackboard. Remove the paper: join PQ, PR. The problem is solved. Note that this experiment brings again into prominence the old truth already noted, that 'Two points fix a straight line'. Thus applied, the old truth leads to a new. Teacher: 'Will you try to put into your own words what we have just learned about the making of angles?' Such attempts form a discipline most excellent for the child. Capacity for generalization, for accuracy, for brevity, for relevant and clear statement, for sound reasoning, self-reliance, and originality—to name the chief aspects only—are all stimulated and developed. In a word, the child learns to express

himself. It goes without saying that the teacher must be patient. Often much polishing and many attempts are needed to get the statement of the truth moulded into shape; yet often, too, the first attempt is a happy surprise and above generous criticism. Of course, the final critic must be the teacher, but first let the class exercise its critical and constructive capacity. In the present instance, the following, with but slight addition, was the statement manufactured :—

Truth : 'If we know the position of the corner and any one point on each arm, we know the size of the angle, and can make another equal to it.'

9. *Further Problems on this Truth for Hand and Eye.*

The above is the well-known principle underlying the use of an ordinary protractor, but it is advisable thus to let the principle be *discovered* and first employed with the utmost simple form of instrument. Too often the complexity of an instrument conceals completely the principle at the basis.

Now give several little problems to produce skill of hand and eye, to further test and give deeper familiarity with the truth just discovered.

Make an angle equal to a given angle under these conditions :—(1) With its corner at a given point; (2) with its corner on a given straight line; (3) with its corner at a given point in a given straight line; (4) with one arm coinciding with a given straight line; (5) satisfying both conditions in (3) and (4). (6) Can an angle be reproduced if we know *any* three points on its boundary?

The fifth problem is not merely to illustrate the truth just won, but to anticipate difficulties in the truths to *be* won regarding the *addition* of angles. This educational principle is, of course, valuable and well known.

10. *Study the Pupil—not the Textbook.*

The teacher cannot, I think, be too cautious in his assumptions respecting the extent and kind of the pupil's knowledge. There appears to be but one way to avoid making incorrect assumptions. We must deliberately experiment. The very worst sin of all is to tell a lad to *get up* at home so many axioms, definitions, or propositions

from a textbook, on which to subsequently examine him in class. This process degenerates so easily into rote-work and cramming. A textbook of science may serve admirably as a logical, rigorous, and systematic presentation of its subject-matter for an *expert*; but, for the *beginner*, is it not quite impossible to predetermine in detail the most effective educational order of development? In mathematical language, the order of presentation should be a function of the individualities of the pupils and the teacher—and of these only. The textbook for beginners is often a superfluity.

Question your pupil; propose to him little problems of a practical nature. Get him to frankly tell you what he already knows about the matter in hand; find out what he can already *do*. For only in firm continuity with his present grasp of things as geometrical can you build enduringly. Hence the following conversation.

11. *A little Educational Experiment: Can Angles be added?*

TEACHER (making two angles, each in a separate position on the blackboard): 'Can we add these two angles, and make one angle equal to them both together?'

PUPILS (unanimously): 'No!' [1]

To prevent misconception of the nature of the problem, it is thrown into other forms, in one instance being asked as a practical problem to be *done*. It soon becomes obvious that the conception of an angle embracing the possibility of the *addition* of angles is new and foreign to the minds of these children. Interesting is it, let me add, to question adults on these same points—adults, I mean, who have received no specific geometrical training at school. I have sometimes received here, too, the same negative reply, and found inability to solve the corresponding simple practical problem. This, of course, merely illustrates an old educational truth. The conceptions of an individual cease to develop with his years after those conceptions have once reached a certain relatively elementary stage, which just suffices to practically interpret his environment. Fuller

[1] It has just struck me that a more successful result might be obtained by proposing the problem with two *paper angles* instead of chalk-drawn angles. The paper-angles can be brought together directly. Experiment only can decide; *in that* lies the teacher's salvation.

development only ensues if the mind is continuously stimulated for a certain period by specific attention directed to these conceptions by other minds in a fresh environment, by new experiments and new problems. Hence, indeed, the rationale of educational machinery, which proposes the creation and the stimulus of a fresh environment of an artificial nature. I say *artificial* deliberately, as a word suggestive here both of bad and of good quality: it is the future task of the educationist to discriminate and allot its due share to each aspect.

12. *Creasing and Turning Experiments to show that Angles can be added.*

I have always found this the central difficulty in developing the angular concept. No amount of mere talking to the child will overcome it: definitions and axioms and postulates, and the rest of the strange crew are but a vain show. Give a problem that demands brain and eye and hand: something (to use Socrates' word) that acts as gadfly to sting the intellect and rouse it from torpor. To grasp a new fact demands the creation of a new thought; gain the new thought, and you are able to interpret other facts. So does thought react on experiment and experiment on thought; neither strictly comes before or after, for together they form new *experience*, and spring together from the old.

First Experiment.—Let each child form a paper angle. *Ask them to make this angle into two angles.* If they can't do it—which will not, I believe, be often—show them how, by simple *creasing* through the vertex. This easy experiment at once puts a different complexion on the matter: obviously an angle can be *divided* into two angles, which together equal the original. Here is a case where division is easier than addition, analysis than synthesis. Not that the two aspects can be really separated, for no sooner is the last fact seen than the other grows obvious too. They are indeed different aspects of one and the same truth; but sometimes, as it were, it is easier to get into a house through the back door than the front.

Let the child cut through the crease, separate the two distinct angles thus formed, finally replace them together. Your difficulty has gone.

Here I would suggest the following extensions of this

problem—apparently very obvious and yet not striking me at the time. Had I thought of them, probably my next difficulties would not have arisen. But prophecy is unsafe with children.

(1) Make the original paper angle into three angles by two creases. (2) Cut the creases and separate the angles. (3) Replace, *in different orders*, to make the original angles.

Repeat these three operations for as many creases as can conveniently be cut, 4, 5, . . . until the children can effect the whole operation with neatness and rapidity, varying the size of the angle throughout until the general truths are clearly seen (1) that any number of angles can be added together; (2) that the order of addition is indifferent.

Perchance the reader who has *not* taught geometry to young children will judge all this to be too ridiculously simple and mere unnecessary trifling. But let him read on and I venture to think he will change his opinion.

Second Experiment.—With wooden protractor placed on blackboard (described on page 30), keep one arm fixed and revolve the other into a marked position; still further, into a second marked position; thus again showing the union of two angles into one. This admirably strengthens the effect of the previous experiments. It is well to keep in mind the two aspects of angular magnitude, direction and rotation.

13. *The Constant Need of Intuitions as Reinforcements to Logic.*

Note, too, the principle of *accumulating evidence* for a new truth. There is a weakness as well as a strength inherent in that species of evidence that reposes solely on rigorous, systematic logic for its establishment. You may assent to each step, yet profoundly distrust the conclusion; because you have no clear view of the whole. The emotion that accompanies genuine conviction has not been roused; therefore the effect of the logic is transitory. This is a pitfall in all mathematical education. It is true that the weakness resides in the individual, and not in the evidence, strictly speaking. But such a retort is irrelevant to an educational standpoint, where the person perceiving the truth, not the truth to be perceived, is the central interest. In education mathematics must rely more upon the fre-

quent introduction of intuitions; both its evidence and educational value can be thereby increased. To bring in once for all at the start all the intuitions (in form of postulates and axioms) upon which to erect a systematic, logically rigorous structure is fatal to genuine progress; experience proves it. Moreover, it is impossible to obtain such a perfect rigorousness; the history of philosophy proves it. Therefore I say, accumulate evidence for your central truths and constantly draw upon fresh intuitions.

Problems.—(1) With use of simple paper protractor add together two given angles chalked on blackboard, or pencilled on a sheet of paper. (2) Add another given angle to these two. (3) Add another to these three, thus adding together as many as four angles.

There was decided difficulty felt in doing these, simple as they look. Remember that the angles to be added are drawn in all kinds of positions, thus :—

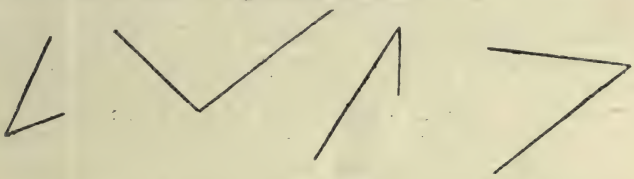

FIG. 18.

The whole operation is really complex, demanding forethought, handiness, and accuracy. There is measurement, transference, fitting together, clear *interpretation* of the work throughout.

14. *Honesty of Deduction from the Experiment: the Truth, and the Truth only, to be stated.*

Beware, my fellow-teacher, lest you unconsciously and incautiously supply the children with a generalization which they have not as yet of themselves perceived and reached. Fruitful, by the way, would be the discussion of the validity of this principle in the teaching of history. Such a blunder I find myself repeatedly committing, so easy a trap is it to fall into. Such a blunder I nearly made here. Fortunately, my general habit is to elicit from the class the appropriate literary expression of the truth received. I write what they give me on the blackboard; then each copies it neatly

into his notebook. Whether it is wholly advisable to do this last I am still doubtful. Now, judge of my astonishment when this statement was given me by one of the class, and acquiesced in by the rest: Truth: 'We see that *four* angles can be added together.' [Refer again to experiments above.] I, in my haste, had been on the point of saying—and what could apparently be more natural?—'Now, you all see that *any number* of angles can be added together.' But *only four!* This caused me to ask: 'Can five angles be added together?' 'No!' was the unanimous answer. As this lesson had already been unduly prolonged, I deemed it inadvisable to tackle this new difficulty—wherein, perhaps, I erred—but simply expressed astonishment at the reply, whereby they might be led to think over the matter subsequently, and be prepared for quite another conclusion next lesson. Is it of supreme importance or not that no deduction should be drawn which the experimental evidence is not clearly seen to warrant?

To ask is to answer the question. Of themselves, young children at home are precise and exact. The accuracy of their deductions from experience, relatively speaking, is remarkable. They are blunt and truthful in statement; able to express themselves where fear has not tamed them. I believe that most children gradually lose these characteristics at school. If this is a fact, why is it so?

15. *The Addition of any Number of Angles really implies the Concept of Infinite Angles.*

I offer a suggestion as to the cause of the children's inability to grasp at once from these experiments the generalization '*any number* of angles can be added together'.

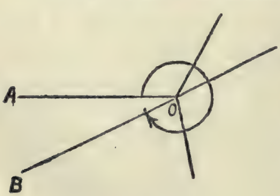

FIG. 19.

We have seen that they succeeded in adding together four angles. Now it chanced that the resulting angle was nearly an angle of revolution or four right angles (Fig. 19).

It is possible, therefore, that there existed a crude and obscure idea in the children's minds that there *was no room left* to add another angle (of the average magnitude of those

taken), since, forsooth, the result might *exceed the whole space round the point, O.* It may be replied : ' But surely they could easily see that a fifth angle, *AOB*, could, at least, be added ! '

To that I would answer : ' Probably the children would not, *at first*, have admitted a whole revolution as an angle '. However, all this is mere conjecture ; the interesting point is that the answer was simply ' No ! ' Note incidentally that already the conception of an angle greater than two right angles was being developed and admitted through this process of simple addition.

If we reflect further that the acceptance of the apparently simple truth that any number of angles [1] can be added involves this other truth, apparently much more difficult to conceive, but coextensive therewith, that an angle is to be a concept embracing the possibility of unlimited or *infinitely large* magnitude (i.e. of any number of revolutions round a point), then we begin, I think, to see why the children instinctively refuse to accept the *general* statement.

This explanation is made additionally plausible by an historical fact. Euclid explicitly avoided the conception of an angle equal to or greater than two right angles. The mode of proof in Book III is sufficient evidence for this, as well as the classification of angles in Book I. It is true that Theorem 33, Book VI, is invalid without the extended conception of an angle of unlimited size ; but we may safely say that such use here was merely implicit, and unwittingly made. As De Morgan says : ' Here the angle breaks prison '. A few experiments with turning movements of a rod round a point would doubtless soon produce the desired extension of thought in a child's mind. Two similar cases have occurred in my teaching experience. (1) Children—and even adults, unfamiliar with geometry—have drawn, readily, triangles and quadrilaterals, but found it impossible without help to construct *pentagons*, believing, indeed, that such figures (bounded by five sides) *do not exist.* Still less could they conceive the possibility of figures bounded by

[1] Strictly speaking, one should say *finite* angles ; for, with the conception of infinitely small angles, the statement above is no longer valid. But, of course, one would hesitate to credit the cleverest child with any such subtle notion as this ; not but that the infinite is ever a difficult conception, whether as the infinitely small or infinitely large.

any number of sides. We must remember that pentagons are figures very or even totally unfamiliar in the child's experience. Four-sided figures (and to a less extent triangles) are quite common. (2) Young children in their early stages of learning arithmetic may obtain considerable expertness in adding, subtracting, and even multiplying small numbers, and yet have the distinct impression *that numbers beyond* 100 *do not exist!*[1] This fact of child psychology may also throw some light on the other two facts, especially the difficulty about adding angles. Be it further recorded that it appears a general truth of the child-mind that it readily extends a principle, once seen to be true for small numbers (e.g. $2 \times 3 = 3 \times 2$, $5 \times 2 = 2 \times 5$, &c.), to all larger numbers familiar to its range of experience, but stoutly refuses to admit these principles to be applicable to *any* numbers (i.e. larger numbers outside its range of experience). Thus a child will often refuse to extend the principle of commutation beyond 100. Of course if the teacher uses his dogmatic authority these effects become masked and even disappear.

16. *Truths—not Theorems.*

It will be noticed that I called our conclusions 'truths'; theorem or proposition were a quite inappropriate name. These imply the existence of systematic logical paraphernalia absolutely unsuited to early work in science. In a later chapter I hope to discuss this point carefully, attempting also to give an approximate account of the psychological processes involved in early geometrical education.

[1] Compare the arithmetical inability of certain races, pp. 349, 350.

CHAPTER V

SOME POINTS IN THE HISTORY OF ARITHMETIC AND THEIR APPLICATION TO THE TEACHING OF CHILDREN.[1]

THIS and the following chapter I propose to devote to an historical sketch of the beginnings of arithmetic and its application to teaching. I shall make use of the fundamental principle which I have mentioned already many times, namely, the parallelism between the development of the individual and that of the race. I think we might almost say that that has become one of the most important and central principles of education, and probably only by an initial development of that principle shall we be able to develop a real science of education. The historians must trace for us the development of the various sciences, and the growth of man's knowledge in the race itself through the different nations. Then the psychologists and teachers themselves, and if possible, where they can be interested in it, the parents, must trace the development of knowledge in the mind of the infant, the child, the boy, the girl, and the adult.

We can practically take it now as established by a large number of lines of evidence coming through many sciences that the individual does recapitulate in his own development the essential lines through which the race has passed —I say the essential lines, not the details. It is, I believe, admitted by experts to be true biologically: it was first found in biology; and now it is seen to be true also for the mental or psychic organization. Of course we have to remember that there are a great many limitations to the principle: there are such things as short cuts, compressions and modifications, both in the biological development and in the psychic development, but the essential truth of

[1] Forming the substance of lectures to Saturday Morning Classes of Teachers.

the position is now established beyond serious attack: future research will probably devote itself rather to discovering its limitations and applications than to questioning its general validity.

We shall see perhaps in mathematics more than in anything else the truth of this principle, because we know the history of mathematics, and, particularly, because we have had vast experience in the teaching of the individual and the human race in mathematical processes.

Unfortunately we do not know very much about the very beginnings of arithmetic. But in all historical researches we can get very definite evidence at certain points, and in the spirit of that evidence we can legitimately interpolate certain other facts, or certain other theories that prove to be facts, just as we do in interpolation in mathematics itself.

We have races still existing on the face of the world who cannot count beyond three, in the sense that they have no numeration, no names for numbers beyond 3, but even a race that is capable of using 3 is a race that has gone very far in the development of arithmetic, when we compare such a knowledge with that of an animal that perhaps, so far as we know, has no conception of numbers. But to some extent we can guess what the nature of the process is. The infant has but a very obscure consciousness of anything at all. It is of course a mistake to think that the infant is born with no experience at all: it is born with a large amount of mental experience, yet its first ideas and sensations will be extremely obscure, and it will not be able to distinguish clearly between its own body and the world around it, and still less between the various parts of its own body. The fundamental feelings will be vaguely concerned with self and not-self. Subsequently these grow more and more distinct. Here we have at once the embryonic ideas of unity and plurality. When I say embryonic ideas I do not mean that the child has a perfect conception of unity, a perfect conception of anything whatsoever. We never get, any of us, a perfect conception, because, as I have often pointed out, every word is capable of an infinite number of meanings according to its context, and it is always developing infinitely, so that we can only speak of the degree to which human beings become capable of understanding the meaning

of words, as of all other things which have a life of their own. We have, as it were, a division of the mental powers of the child in two directions: the perception through the senses dealing with the concrete—so many fingers on its hand, for example, so many toes on its feet—and the corresponding parallel developments of conception. We have gradually to deal with conception and abstraction on the one side, with perception of the concrete on the other: there with ideas of unity and plurality, and here on the concrete side with the actual objects from which they have been obtained, by the help of the intellect itself, for example, so many fingers, so many stones, &c. I would remark here that we should take care, in the teaching of arithmetic to infants, not to introduce big numbers too early; a big number in itself is a difficulty; and this again is one of those cases where we get a close parallelism between the individual and the race. It has generally been a difficulty with the race, and it is generally a difficulty nowadays with children. When we speak about numbers, unity, and plurality, we find, as a matter of fact, that our ideas of the many are generally limited to the finite many, and even these are limited indeed. Unity, the many, the infinite—how many of us can deal at all effectively with the infinite in Mathematics? It is generally beyond the capacity of any one who has not made a special study of it.

Now, I shall deal with the main stages of development in arithmetical symbolism and arithmetical ideas. I say the main stages of development, because the more we think about these things psychologically, and in our experience of teaching, the more stages we find the child unconsciously passes through. I have repeatedly said how apt we are, over and over again, to assume too much in the minds of our pupils: the number of stages that their minds really go through is almost beyond belief: I am only, therefore, attempting to deal with the main stages I have myself observed.

(1) We have first the obscure mental stages. What these precisely are I do not think any one knows. But we get gradually a distinction between self and not-self, and self itself splits up into fingers and toes and so on: not-self into the outside world, so that we gradually get the idea, the embryonic idea, of unity, and thus it becomes possible at last for an infant to draw a conclusion like this: This finger plus

that finger equals two fingers. Of course our common sense tells us that the infant does not put it in a form like that: it is not a conscious but essentially a subconscious process that at first takes place; but the infant is at any rate able to deal finally with the consciousness of a truth like this—that here are two distinct things apart from myself, two fingers, two people in the room. (2) The next stage—also without language—we have now to consider. There was a controversy for a long time, taken part in by the celebrated Max Müller, whether there could be thinking without language. It is seen now that *most* of our thinking is done without language. We recognize fully now the subconscious life. I would liken it to an enormous ocean, and consciousness is simply the surface of that ocean, and most of our thinking is done in that ocean of subconscious experience. There is no doubt that the infant and the child do a great amount of subconscious thinking without language at all. The second stage will be, then, for example, that two fingers plus three fingers = five fingers. These are only, of course, types of the kind of thinking that is done. (3) Perhaps another stage will be—I will not venture to put the years at which this takes place, because it will vary with different children—two cows plus three dogs = five animals. This is an application of number to heterogeneous things. When a child catches the idea of things, say five things, it has taken a large step in conceptual progress, however vague. (4) The next stage, then, would be, two things plus three things = five things. That is still more general. The process becomes more and more general, more and more abstract as the child goes on in its development, but still it is, relatively speaking, highly concrete, because it refers to concrete objects. (5) We might take as the next stage, two plus three = five. Of course when the child has got to that point it has reached a very advanced stage in arithmetical operations: i.e. when it can say two and three are five, dealing with abstract numbers apart from any particular experience with which they are associated at the moment.

Of course the concepts of any of us really spring ultimately from the action of the mind upon a number of single experiences, and never does the concept become entirely free from the kind of experience that developed it, and the

wealth of the concept is proportional to the amount of concrete experience of things that we can see and touch, &c. I venture to say that even in a Newton the concept would never become absolutely and entirely free in his mind from the concrete experiences by which it was gained.

After these stages have been gone through, or perhaps simultaneously—it depends upon the child—sometimes simultaneously and sometimes subsequently—we shall get all these stages over again in *spoken symbols*, for so far the process has been largely subconscious; but now we get it all over again with spoken language, and directly we get spoken language we get a highly symbolical form. Notice the exceeding abstractness with which even children can deal at last.

The process then goes all over again in *written* symbols with still further abstraction, viz., with the special symbols of arithmetic, the marvel of which is concealed from us by our intense familiarity with them. When we come to 'two plus three = five' we get on to the beginnings of Algebra. It was once asked, 'What is the use of Algebra to people?' It is possible that Algebra in its more complex stages was not of much use to the questioner; but it was the asking of the question that showed a complete misconception of what Algebra is. Algebra is not merely manipulation of written symbols, but Algebra is in its essence generalized Arithmetic. Any child who can use intelligently the fact that three and two make five is grasping already the fundamental spirit of Algebra. The elements of Algebra begin directly we begin to generalize about numbers. When we say that 7 times 5 is equal to 5 times 7 that is already embryonic Algebra; but when we say that the order of multiplication of numbers is indifferent, we have got an extremely broad idea of Algebraical operations, apart entirely from its expression in literal symbols. Where Algebra begins and Arithmetic ends no one has ever been able to say yet, simply because the passage is continuous from one to the other. The next development (6) would be $2a$ plus $3a = 5a$, where a is a finite whole number. [Note that our truths require reconsideration and modification when *infinites* are introduced—whether small or large.] A still further generalization is $2a$ plus $3a = 5a$, where a is a finite number, whole or fractional: then $2a$ plus $3a = 5a$, where a is a number whole or fractional, positive or

negative; and a still further advance in generalization is $2a$ plus $3a = 5a$, where a is a finite number, rational or irrational, imaginary or real. Then we pass on to $ma + na = (m+n)\, a$, and so on. There is, indeed, no end to the increasing generalization and degree of abstractness. What I wish to bring out is the very large number of stages which the mind goes through.

The language of the Veddas of Ceylon is not sufficient to count beyond half a dozen. We can see how that comes about. They have special spoken words for 1 and 2, but if they want to say 3 they have to repeat the word for 1 together with the word for 2; and so on. Thus, one = ekkamai, two = dekkamai, one more = otameekai. Thus the names for large numbers is prohibitive of general use. Again we find, in a certain Australian tribe, the height of spoken symbolism reached by four and five, whose names signify many and very many respectively. There are many nations on the face of the earth very limited in their use of numerical language: some are limited to numbers up to 5, some to 10, some to 20, and so on. The Greeks themselves, except the most able professional mathematicians, never went beyond 9,999,999: number was simply infinite after that. Now although many tribes had no words for higher numbers, it does not necessarily follow that they could not deal with higher numbers, because here again comes in the partially conscious but predominantly subconscious process of dealing with things that have no special spoken signs for them. Some of our greatest artistic work is of this kind—predominantly subconscious; you cannot put it into language at all: any one who can put any part of it into language makes thereby a discovery in scientific development, for the progress of science depends upon the ability to draw into the focus of linguistic consciousness that which we have reached sub-consciously. How did these nations do if they wanted to apply arithmetic—not in language—to large numbers of objects? They did it easily in some such way as this: 5 fingers equalled 5; that was symbolical: 2 hands held up equalled 10; and if they wanted 20 they got a couple of men to stand together with their hands up: 3 men standing together equalled 30; and so on. By this method they could count large numbers of sheep &c. in flocks, and they could carry on their commercial

business by means of this process. The Chinese to this day carry on some of their most difficult commercial transactions by means of this dumb kind of symbolism. Putting it shortly, their system is this: take the fingers—the little finger first—1 is the first joint on the outside: 2 if we touch the next joint: 3, the third joint on the outside: 4, the lowest joint on the front: and so on upwards to 6, and finally down the inside to 9. In that way the little finger is sufficient to symbolize the first nine numbers, and then we have only to multiply by 10 to get 10, 20, . . . to 90 represented similarly by the second finger. We get hundreds on the third finger, thousands on the index finger, ten thousands on the thumb, and so on. The Chinese calculate so extremely rapidly in this way that by touching hands underneath a cloak they can make bargains without any one knowing anything about it.

Now, this is a fact I would very much like to impress upon all teachers, the deep importance of this subconscious life without language at all. I think I might, repeating a previous simile, partly typify it in this way—imagine our total psychic life to be a vast ocean which is constantly being deepened by streams poured into it from all the senses, the two eyes, the ears, the fingers, the muscular senses, and so on. We might say that for mathematics the most important of all are the touch and the muscular sensations, and this is typified by the fact that we have ten fingers and numberless joints, but only two eyes, and two ears. Our pupils may be doing mathematics when they are neither speaking nor writing. The fact that many races have had difficulties in getting spoken symbols for high numbers is not to be wondered at. We ourselves are limited, for which of us can give a name to $10^{10^{10}}$ for example? We can write down the symbols in Algebra, but we cannot read it in the ordinary language of numeration.

What then are the main stages in the development of our notation? And this is the point, and a specially interesting point, which I want to bring out. How did the human race arrive at this system of notation? Notice this extraordinary result. We are so accustomed to it that we rarely appreciate sufficiently the fact that with the ten symbols, 1, 2, 3, 4, &c., and 0, we can write any number, however large. Yet the invention is really a marvel, and we can only see the depth of it when we investi-

gate the process by which it was developed, and consider the thousands of years that it took to develop it, and the large number of nations that took part in the development. Throughout let us not forget the application of this to the teaching of children. In the teaching of children, which of these stages can I omit ?—is always an important and difficult question. This will depend upon the children. I have divided the stages into certain groups, under certain principles, but there is so great a number that we can put in but the main ones, and these appear to be as follows :—

(1) We have distinct individual units. Many races still to this day represent 5 by means of fingers, or by 5 stones, or it may be by 5 rods. But notice this, that these things are undoubtedly in some degree symbolical and abstract because the fingers do not necessarily stand for fingers; they may stand for sheep, they may stand for men; and although they are highly concrete in the sense that we can see them and touch them and handle them, that we can apply to them our muscular senses, and our tactual and visual senses, yet they are symbolical in the sense that they stand for something else than what they themselves are. For example, 5 fingers may stand for 5 sheep, so that here again there is practically no stage at which a thing is absolutely concrete, nor any stage at which a thing is absolutely abstract : there is no experience that does not contain both of these elements. If there is, however, an excess of the abstract, children will not understand it ; and yet we cannot separate the two wholly ; and those who speak of wholly concrete experience or wholly abstract experience are surely talking about something about which they have not thought sufficiently. The important question is—the degree to which the experience is concrete and the degree to which it is abstract.

Thus we come to the next stage, (2), where we have groups of collective units—and this time we have made a very large advance in the history of mathematics. Some nations have 20, that is the hands and the toes ; some nations have 15, some 10, as for example a whole man, or, if you like, two hands, only this time you must think of the two hands as forming a unity, and this can be used as a unity for something higher. We might have 20 as two heaps of stones with merely the stones as individual but grouped together in each

heap and forming a higher unit (10). In a great many races they heap the stones up, and when they have got up to ten they put a new heap. Some races could reach the first stage but not the second. How many thousands of years it would take to develop the one from the other I will not venture to say.

(3) We now come to a higher stage. Take a shilling: it is a concrete enough object, certainly. But it involves a high degree of abstract symbolism because it stands for something quite other than itself. The shilling—look what a complicated thing a shilling is: it stands for twelve distinct pence. In the first two stages we can see and handle the individual things in them; but here we cannot see or handle the individual things (the twelve pennies the shilling represents). The shilling then is more symbolical, it stands for twelve distinct, invisible, intangible objects.

These are the three main stages. There is just the same process in the development of all the collective units, for example, the hour glass as the collective time measurer, the foot rule as the collective space measurer, money as the collective value measurer, and the avoirdupois pound as the collective weight measurer. These are enormous advances in mathematical development.

Secondly, we come to the evolution of the more complex machinery for counting. Here are two things that have to be done, and one is about as difficult as the other to the human race. How did they do the counting? There are two parallel developments—how to do the counting and how to record it. We find throughout the world, in almost all nations, ancient and modern,—the Chinese, the Japanese, the Egyptians, the Greeks, the Romans, and so on—a machinery for this purpose, viz. the machinery of the apparatus that we still use to a great extent in our kindergarten—the abacus. In some places, instead of being wire-rods like those on the abacus, the machinery is a board with chalk marks, or perhaps sand ruled into columns, but the principle is precisely what we have in the abacus. Notice how geometry of position comes in here to the help of arithmetic: geometrical conceptions enter into and support the veriest elements of counting. If we put our units on the first wire of our abacus or it might be in the first column, then they

represent 1's; but if we put the units in the second column or on the second wire then they represent a higher order, they represent 10's; put them on the third wire and they represent 100's; and so on. As a matter of fact, if it were not for multiplication and division there would probably never have been any development of mathematics further than this, because addition and subtraction can be done with reasonable practice more quickly with the abacus than with our modern symbols. Now, it has been suggested that not only did geometry of position suggest this, but also position in a social sense, difference of castes. The men of high rank would sit on high benches, and the men of higher rank would sit on higher benches still.

The Graphical Abacus.

What is the next stage? After this first has been used for thousands of years the next stage is that gradually a number of clever people ask—I will not say *one* clever person —What is the use of a complicated instrument like this? Why not have a symbolic arrangement of it? And if it happens that in that nation there also exist some simple symbols for the numbers, and these ideas coalesce, there emerges a remarkable result. For example, suppose the number 'one' is represented by a dash, or a circle, or a dot, or some simple sign; these are the fundamental ways of dealing with 1. Now put the two ideas together, viz., symbols for numbers and position to indicate value derived from the abacus, and we obtain what, in its later stages, has been called a graphical abacus, where blank columns replace wires. Then we put some sign in this first blank column for the units. If we use the written symbols we put the 1's in the first column, the 10's in the second column, &c.

M	C	X	I
5		4	3

If, therefore, I want to write here 5043, I put the 5 in the fourth column, that is, five thousand: I do not put anything in the third column as there are no hundreds—mark this carefully; then I put 4 in the second column, and 3

finally in the first column. I have used the Arabic (originally Hindu) symbols for the number, and the Roman to indicate the values of the columns; because in the Middle Ages they mixed the two—Roman and Arabic—together. This is a frequent occurrence—the struggle for existence amongst sets of symbols. We still have Roman figures on clock faces. This, then, is the next process, where, observe, the symbols are becoming more and more abstract while gaining in power and meaning.

Evolution of the Zero.

What comes after that? Just as we displace the rods by columns, so we can remove the columns, on the understanding that difference of position implies difference in value or units; but if we rub out the columns there is one thing we must remember, and that is, that some of our spaces are occasionally *empty*. Now the Arabic word for an empty space is 'sifr', our 'cipher' (which, also, in another form, is, curiously enough, *zero*). If, therefore, we are going to rub these columns out, we must make some mark to show that there is an empty space. The simplest one we can make is a dot; and that is what happened. Rub out the columns, put dots for empty spaces, and then there emerges 5·43, meaning our present 5043. Notice that the dot does not represent a number, it is simply a contraction for 'here is an empty space'. It was several hundred years before this use of the dot came about.

What happened after this? These symbols were used far more in commerce than in anything else, especially among the Italians, and unfortunately—but perhaps for us fortunately—I am only giving here one mode of development of zero—merchants were tempted to cheat, and occasionally they used to rub these dots out. In order to stop that, one had to make a bigger mark. The next thing, therefore, was to make a small triangle, but it was found that this also was occasionally rubbed out, and thus there emerged a pretty large polygon, which may be seen in some old books. If we make a polygon quickly we get our modern round or elliptical form of zero[1]; and

[1] Just as has happened in musical notation, where convenience of writing has gradually substituted the round form for the square or the diamond.

the last stage is thus reached as far as the form is concerned; not so with the interpretation. For when we have once got our zero in that shape we think of it not merely as an empty space; people began to regard it as itself a number though it is really the denial of number, and our *numbers* are now 0 1 2 3 4 5 6 7 8 9. Here then we have reached, through centuries, nay thousands of years of trial and failure and success, our marvellous development of number-symbols.

Educational Application.

I have stated already, it is not a question of the stages that an individual *can* be made to pass through, but of those which he *must* pass through, if he is to enter into the full possessions of his race. We can cut short the process or lengthen it, but we cannot cut it off altogether without arresting the individual's development. The two extremes into which one may fall in dealing with education are—(1) to try to cut the ancestral process too short, (2) to spin out certain stages of it too long. What is the result? If we try to cut it too short, to jump, for example, in arithmetic over too many of these processes, then the history of education shows that what happens is that our pupil simply gets rote-work knowledge. It is true his memory is disciplined, but almost everything else is left out of account; and his rote-work knowledge is not applicable to an interpretation of new experiences: he becomes dwarfed in intellect. That is an extreme that has been developed over and over again in mathematical teaching. When we start a boy in geometry with Euclid or any equivalent formal system, we cut the ancestral process too short. In the days before kindergartens, when we began to teach children by giving symbols straight off—there again was an attempt to cut short the process: the aim was sensible enough, to get the child on as rapidly as possible to the higher abstract and powerful acquisitions of the race, but the mistake lay in not going through a sufficient number of intermediate stages, without which the child could not intelligently grasp these acquisitions. Of course largely different individuals will require largely different treatment. What may be a reasonable ancestral 'short cut' for one set of pupils may be fatal to the progress of another set.

The other extreme is—we go from one extreme to another generally—to spin out the ancestral process too long, to immerse the child so much in sense experience that the intellectual part, the abstract, the reasoning part, does not get sufficiently developed. The result of that is that the child's mind becomes full of incongruous masses of undigested and unrationalized sense experience. Which extreme is the worse I will not venture to say. One extreme is illustrated where the kindergarten process is too long and the other is typified by the old Euclidean teaching in its worser forms.

The application of my thesis to the earliest teaching of infants in arithmetic is obvious. Let us rather consider the use of the abacus in the later stages. Now, broadly speaking, there are five well-marked stages—some children would require more, some children would require less, and it would of course depend also on the capacity of the teacher. The most perfect teacher is he who can put the child through the minimum number of ancestral stages of the development of knowledge consistently with the child understanding the matter. In the first place our aim is to get the child to understand the place-value of zero.

The first step would be something corresponding to the abacus: it might be small stones or other single objects arranged in heaps in certain positions, which are just as good as the abacus; but the essential idea must be there, namely, that we have distinct objects, the numerical objects, put into different places to represent the values of different units; with this would go the spoken symbols or numbers.

The next stage would be the written symbols.

The third stage would be the paper or graphic abacus—as we have it now.[1] The fourth stage some teachers might find it advisable to omit, and that would be the dot, viz., a special symbol to represent an empty place in anticipation of the zero.

[1] A particularly substantial amount of time and attention should be devoted to this stage. Even at much later periods return may, with great advantage, be made to the *graphic abacus*, so centrally important is the principle of place-value—irrespective of the zero—in the science and art of Arithmetic. I have found by experience that the usual difficulties in teaching 'long division' almost disappear when this treatment is adopted.

The last and fifth stage would be the full zero itself. These are well-marked stages that most children should go through. I would also give the children themselves every chance of finding modes of dealing with difficulties. I do not say that we can get many children to do this; perhaps only a small number will be able to suggest the idea of place-value, but we can get some to do it; and experience shows we can get some children to invent a symbol for an empty space: give them indeed every chance of finding symbols.

Here is an interesting historical fact. About fifty years ago in a school in the West of England there was a mode of teaching the meaning of zero and place value for children which was, curiously enough, almost identical with these stages that I have traced above; that is to say, the teacher went through the whole set of processes from the real abacus through the graphic abacus, and then to the final system as we have it nowadays. But oddly enough, on investigation it was discovered that she was an excellent kindergarten teacher who knew nothing about the history of mathematics, but simply felt that this was the inevitably best mode of procedure dictated by her experience. Through her own skill she had invented this mode, this process which is exactly the same as would have been suggested by any competent teacher who knew the history of mathematics. When this came to my knowledge I made investigations amongst teachers, and in not a single instance did I find a case where the invention was suggested by a knowledge of the historical development of arithmetic or other branches of mathematics. All these teachers had invented methods after much error and trouble which could have been saved had they known the outlines of the history of mathematics, and in my opinion this fact serves to bring out the value of our principle for teachers. We want teachers, as I said, to investigate the mental experiences and processes that children go through. We must investigate the psychology of the child, the psychology of boys and girls, the psychology of students of both sexes; and if teachers would only put down in language—clear language—their experience, they would help us to make great advances in the science and art of education.

CHAPTER VI

THE NATURE OF THE PROCESS BY WHICH NUMBER IS DEVELOPED IN THE RACE AND IN THE CHILD.[1]

The Development of the Number System, through the Abacus, and onwards.

I SHALL analyse, so far as possible, these stages of development in accordance with the diagram (see Fig. 20, and footnote 2 below for interpretation), and try to find out in what way the race has developed arithmetic. I would also ask teachers to verify by their own observation and thought the thesis that the same diagram that we use to illustrate the development of knowledge in humanity would also represent the development of knowledge in the individual. An illustrated diagram is of course only an illustrated diagram and does not represent every phase of the question; it can only draw into prominence perhaps the fundamental aspect of it; and one does not wish more to be drawn from the diagram than is intended. Every diagram of this sort is partial.

We shall start with the central disc.[2] What is contained in that circle?

[1] Continuation of substance of Lectures to Saturday Morning Classes of Teachers.

[2] To avoid confusion, the reader will note that the central disc is white, the first ring (very narrow) surrounding it is shaded, the next ring (forming the second) is white, the third ring is again shaded, the fourth ring is again white, and so on alternately. The shaded rings, symbolizing *thought* activity, proceeding from within outwards, become gradually broader, while the white rings, symbolizing sensation, become gradually narrower, until at the outside the shaded ring is very broad and the white ring has almost disappeared.

Well, this innermost circle represents the origin of dealings with number—the very simplest elements of all. Now although it is predominantly white, there are, or ought to be,

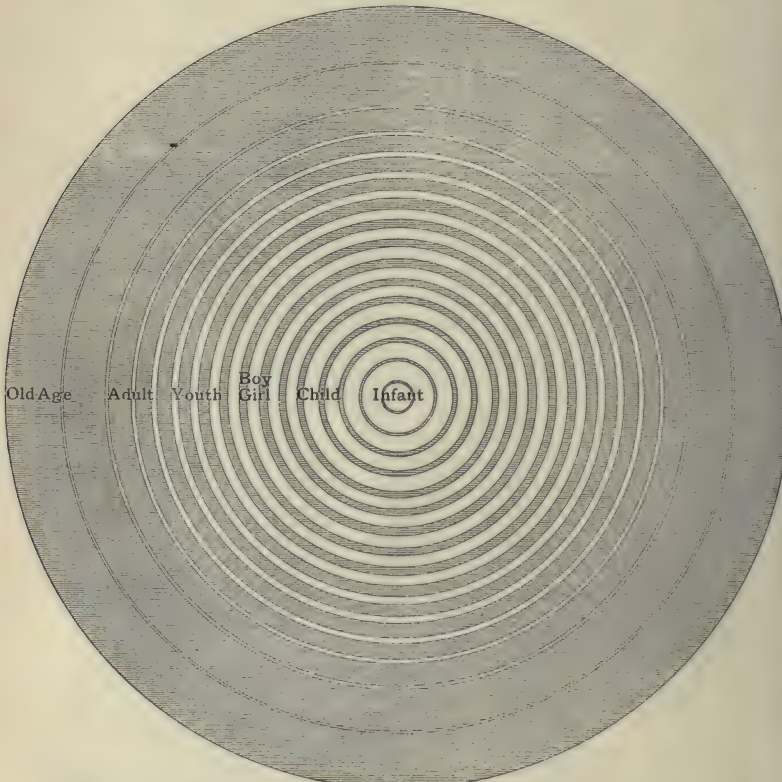

FIG. 20.

a certain number of shaded spots in it, illustrating this fact, that when we begin to analyse deeply any kind of experience, whether of the infant or the earliest beginnings of knowledge and experience in the race, we never find a case where there is not some element of thought activity. That is an extremely

important thing to remember. We appear never to get a piece of experience either in the child or in the race that is wholly sense experience. Where the element of thought creation or thought activity first comes in we have never been able to discover. Some philosophers say that it does come in later at a certain point—the empirical school; some say that it exists in a rudimentary form at the very beginning—the idealists. But the absolute origins of things do not concern us: it is indeed questionable whether we can ever discern them; what we are concerned with as practical teachers is the actual condition of the experience at the stage when we can begin to observe it with reasonable certainty.

The first very narrow ring (shaded), represents thought. Think of a people that is able to represent objects by other objects. Instead of counting the number of sheep, for example, or rather, instead of keeping a record of the number of sheep by having the sheep themselves, they have a record by perhaps a heap of stones equivalent in number. Now there are certain elements created by the mind itself in the very idea that one object is to represent another. In the outside world there are no ideas: clearly there is no symbolic representation of one thing by another; that must come from the mind itself. How it comes we are not concerned with. Man is capable of producing it by certain creative functions of the mind; we say by the use of attention, memory, imagination, analysis, synthesis, and so on. All these mental activities enable us to produce the idea that we are going to represent one object by another. That is not in nature, that is in the mind, it is contributed by the mind. We shall therefore imagine written on this first shaded ring as the contribution of thought, stimulated, of course, by the sense of dealing with objects around it, simply this: 'A mental creation of the idea that one object is to be represented by another'. Now, I have analysed the thought in this process, but, as a matter of fact, in actual experience the thought and the product, as applied to experience, go together in the mind. The practical application of the thought or idea is, that so many stones, or sticks, are actually used for example, for so many sheep, or men; so that the second ring will be white. And here we have the application of the idea to the concrete. But remember they are not sticks as sticks, they

are sticks that represent other objects, they are numerical sticks in other words. At the same time these numerical sticks, though symbols, are themselves objects we can handle and touch: we cannot handle and touch the idea; but we can apply it to these things. We might call this first process a process of symbolization.

What about our next ring? The next discovery would be that we need not put 10 sticks together to represent 10 objects, but that we could put, for example, one big stick to represent 10 small ones. The contribution of the mind there is clearly, again, something that we do not get in nature itself. We look at an object with a certain definite purpose, and say, I will make one of these objects represent ten of some other object. That is a further degree of symbolization: it is more complex; it is simple enough, but it is not a simple thing to the infant mind; and it is not a simple thing to the race in its beginnings of arithmetic. I have already pointed out that there are races still existing who cannot represent numbers greater than 3.

The third ring will therefore be shaded, symbolizing the idea of a 'compound unit'. Because our contribution is a contribution of thought it implies a higher degree of symbolization—this idea of a compound unit. It is also an economy of thought. It is a higher degree of abstraction, further symbolization, further thought, or complex conception. What is the result as applied to the real concrete world around us? We have to apply the idea to real concrete things, things that appeal to sense-perception. So the fourth circle will be white, the application of that idea to the concrete world. Thus, the hand, for example, may be taken as representing 5, not thought of simply as four fingers and a thumb, but as a symbol representing 5; and a man may be taken as representing 10, or, as in the case of the Egyptians, a man holding up his hands in astonishment may represent a million. But notice here that even this stage, the *picture* of a man holding up his hands in astonishment to represent a million, is a further stage than the one we have got to, because I have only got so far to the stage that a concrete man represents a certain definite number, say 10, a compound number it is true, but a very concrete and real number, because we can see him, touch him, speak to him. By a similar evolution of thought, a shilling will represent 12 pennies—

that is an application of the previous idea. Of course the idea and the application of it will probably spring forth together from the brain. We are now simply analysing the chemical compound, as it were, to see what experience the mind is moving in. I will just again note here that the symbols are highly concrete, being visible and tangible. The fourth ring, then, on our diagram, is the application of the idea again to the concrete world of sense perception, and the symbols are still highly concrete, for they are both visible and tangible, and, in the case of the man, if we forget what he stands for *he can tell us*! We, therefore, draw our ring unshaded, or white.

The next stage is again one of creative thought-activity. What is the idea this time? The idea, symbolized by the fifth ring (shaded), is that it is very inconvenient to take a man about with us to represent certain objects, or even to take stones and sticks; but if we make a mark on a piece of paper and take that with us it is much more convenient. Again, that is not in nature; it is a creation of the mind; we may call it the economic tendency of thought, the tendency of thought to save itself trouble. The tendency of every schoolboy to save himself trouble in mental things is one that may be objectionable from some points of view, but it is an extremely good impulse to work upon, and it is possible to get your pupils to invent excellent symbols by this very tendency to save oneself trouble in thinking.

Mathematics to a very large extent indeed is simply the creation of devices that save us trouble in thinking. Take the case of a^n. Unfortunately, in introducing this new symbol, it is the custom of too many teachers simply to state to the pupil that a^n stands for $a \times a \times a, \ldots\ldots n$ times. Now, I should say that although this mode of presentation seems to economize our own time and the time of education, yet it does not do so in the long run. If we give a contraction to the pupil before he feels the need of it, then I think we are wasting time, because we have to go through that process somewhere or other. Look at the development of the index in the history of Mathematics—I shall speak more about that in detail afterwards; but as a matter of fact we find that Algebra went through a large number of stages before it reached a highly-developed form like this. First of all it was purely literary, in the

sense that there were no spoken symbols at all: it was just like Geometry; and every truth was put down in ordinary language. If we go back to the earlier history of the Hindus, the Arabians, and even in the Middle Ages, a great deal of Algebra was simply of this nature—non-symbolic. They stated a truth like this: if the difference of two squares be divided by the difference of the numbers, then the result is that we obtain the sum of the numbers—simply a piece of ordinary English. Then we find that mathematicians got tired of these long phrases, so they contracted, and we then get what are called syncopated symbols. The first stage is what is called a rhetorical stage; and then you get a syncopated Algebra: go a little further and at last there emerges modern symbolical Algebra. But if we look at the works of mathematicians in different ages, we shall probably find, say in the year 1500, a German, or a French, or an English mathematician writing *aaaaaa* where he means a^6; and that was done for centuries. Then some bright spirit suggests that it would be convenient in saving time and space to insert the number six in some such way as to suggest that six a's have to be multiplied together, and there were at least a dozen suggestions as to how to do it. One proposed to do it this way—$6x$. There were all sorts of ways of doing it, till at last Descartes, improving on the suggestions of others, hit upon the method we have got now: a^6. What I want to bring out is, that we find even Newton using the old and new styles on the same page. We do not want to take five or six hundred years to do this or that, but we do want to imitate the spirit of the development, and the application here is:—do not give contractions—and this applies to the whole science—do not give contractions until the pupils feel that it is a saving of time, until they are quite sick of using the old methods; and we shall find that if we judiciously develop in this gradual way the demand for economy of thought in our pupils it will have its application all through their mathematical development, and perhaps in other branches of thought, and finally lead to economy of work.

To come back to our diagram. What is the next development? We have got certain symbols, the man, the pennies, and so on. The next development would be that we

want to have a symbol that we could write down on paper and put in our pockets. That is a further contraction, and it is further removed from the concrete because this time the symbol is certainly not tangible, it is only visible; so the application would be the discovery of written numerical symbols. Here, then, we have already got a highly complex piece of symbolism. The sense perception appealed to is sight, not touch: touch is out of count now. Then, what comes? The discovery that if we want to represent a very large number our symbols get unduly numerous—we cannot remember them. Here recurs the need for economy of thinking. Look at the Egyptians—to represent such a number as 21,232,378—which we do with eight symbols—look what they did.

20,000,000 = Two distinct representations of the symbol for ten millions.
1,000,000 = The picture of a man holding up his hands in astonishment.
200,000 = Two representations of the burbot (representing 100,000).
30,000 = Three representations of the pointing finger (representing 10,000), &c.
70 = Seven representations of the symbol for 10.
8 = Eight representations of the upright staff (representing unity).

Altogether they used twenty-eight symbols for this particular number. (See diagram, page 246.)

We see the precise nature of this development. First the staff, then the picture of the staff, then what we have nowadays. If we want to write numbers after that there is no end to the number of symbols that we have to invent. Then comes the extraordinary discovery that we should indicate value by relative position. This then is a shaded ring. It is not known where that idea came from: it may have been taken from the idea of social rank; but it is certainly again a contribution of thought—it is not in nature, it is an idea of the human race, it is applying to numbers the principle that relative positions shall represent different values, and thus we get our system of place-value. We find now, to our astonishment, that nine different

symbols—and less if we take a different radix of notation—are sufficient to represent any number whatsoever. Certainly they may have to be repeated.

Now, the application of that idea, through sense perception, to the external world, is the discovery of the *abacus*. That is where we get our place-value—each different row represents ten times the value of the preceding row.

This gives us another white ring, and then comes another shaded ring—representing the idea of abandoning these highly concrete symbols and using pictures of them instead; thus we develop gradually written symbols wherein the original ancestor has almost entirely vanished.

So the process goes on without end. We have the graphical abacus, and the various stages that that went through, and the discovery of zero; and so on. Thus is evolved an increasingly complex abstract symbolism, and at the same time greater power of interpreting thereby the concrete world around us.

Application to Education.

Now apply this to the individual. We have seen what the race has gone through. The individual goes through the same process, only much more rapidly. In his bodily organization he goes through the same main stages as the race, but much more rapidly, and so he does in his mental organization. I have said that I do not think there is any more fundamental principle at the root of education than that; and it is possible in the next generation of teachers that we shall get applications of the principle that many of us now do not dream of. There is no end to its applicability.

Applying this to the individual, we have the still wider problem—to investigate in detail the development of mathematics in response to the development of particular *occupations*. And then we have to apply the results to the development of mathematics in our schools. It would take perhaps a whole generation to work this problem out. One sees the beginnings of it already, more in America and on the Continent than here. How, to put it broadly, are we going to find out what boys and girls are most suited for, what their talents are, and therefore what, broadly speaking, their future occupations should be, and how to develop

these talents, and yet at the same time preserve a liberal culture—that is the great problem we are now working out.

Applying our method—looking at our diagram—I have marked in the centre the word 'infant' to illustrate the fact that, in the infant experience, sense-perception is vastly predominant. Looking at the question from a physiological standpoint, the difference is this, that in sense-perception the stimulus to activity comes mainly from the outside world, the stimulus acts upon the nerves mainly on the surface of the body; whatever nerves they may be, it acts upon those at the surface; we may say therefore that sense perceptions are in the main peripherally stimulated. On the other hand, in thought activity, the direct stimulus arises wholly from within the brain itself, so that thought activity is centrally stimulated. Now, that distinction is of course an essential, remarkable, and important distinction; and if we consider the difference between these two elements of experience in other respects, we recognize its importance still further. We may think, for example, of the difference in vividness. When the stimulus is external there is a vivid sense-perception—I see the object, I handle it. But if I shut my eyes and try to reproduce it, I can only conjure up a memory-image which is at the best slurred and indistinct. And here again we see the importance for the child of the external stimulus, and of the sense-perception, of repeated concrete external application so that it can verify its thought and its memory. Again, we can look at it in this way. Take ourselves as grown-up people: if we had not had repeated verification by sense-perception, we should of course be extremely ignorant; and it is a hundred times more so in the case of the infant. Physiologically then, the adult is in the main centrally stimulated, and the infant is in the main peripherally stimulated.

On the other hand, to avoid misapprehension, I would remark that it is a vast mistake to suppose that the infant experiences nothing but sense-perception. The infant exercises a considerable amount of thought activity; were it not so the mind would soon become a blank: no systematic knowledge or power to deal with the world would ever be produced at all. For all tested experience shows that the

development of the mind is a progressive and continuous process. The fact is very frequently forgotten in the teaching of infants in the kindergarten, so much forgotten that a considerable number of capable critics do not believe in the system, and in Germany the kindergarten as we know it has in great measure been abandoned. Perhaps it has been overdone. Possibly we have forgotten too much that the child does think as well as use his senses. There is a tolerably rapid development of the purely internal mental activity, as shown in memory, in classification, in analysis, in attention, in abstraction.

Thus we pass on to the boy or the girl, in whom, roughly speaking, we find that the sense activity is about half the mental experience: and so to the youth, in whom the thought activity now begins to predominate, but with still a considerable amount of sense-experience; and at last we reach maturity and old age—the other extreme. We know that even in very old age in special cases there may be an extremely vivid mental inner life, consisting of memories.

The development is throughout continuous; but we are apt to be deluded by the great predominance of sense-perception in the infant, and the relative insignificance of sense-perception in the old man: we are apt to be deluded by the idea that experience is all sense-perception in the child, and that it is all thought in the old man. That is a fatal mistake for education.

Here again I wish to suggest a caution. Consider that well-known maxim, that teachers should proceed from the concrete to the abstract. Now can we point to any part of the diagram representing the development of the human being, where it is not possible to claim that we are proceeding from the abstract to the concrete at one moment, and the next moment in exactly the reverse order—viz. from the concrete to the abstract. The truth appears to be that *all development grouped into large periods proceeds from a relatively high degree of concreteness to a relatively high degree of abstractness.* Taken in the original crude form the above maxim surely misleads us.

Then, as regards the value or the function of definition—Think of any conception, mathematical for example, say a triangle—What does it represent to us? What arises in

the memory when we see the word 'triangle'? It will wholly *depend on the amount of sense-experience* we have had of a triangle: a mere definition of it means nothing. But if we have seen and handled a triangle, in concrete form, if we have seen it in numberless forms and positions—perhaps helped to build bridges with it—calculated its area, &c., it means something; and it will mean so much the more, the more concrete experience we have had of it, and the concept is valuable in proportion to the degree of well-ordered systematized sense-experience it calls up. To some it will be poor, but the amount of their experience with it will have been poor, and to others it will be rich, and their experience will have been rich. Think of the words 'number', 'quantity', 'line', &c., these mean nothing unless the mind has dealt with the sense-experience that originally gave rise to them. It is well established that every thought we have, every feeling we have, every internal mental state, is accompanied by a certain amount of muscular activity, either voluntary or involuntary, conscious or unconscious, but mostly unconscious. That is an interesting and instructive truth. It is quite possible for a person to exhaust his or her muscles by thinking a great deal—indeed to exhaust to some degree the muscles all over the body by sufficiently hard thinking. We see, then, that the question, in teaching, is not so much whether a piece of knowledge is abstract or concrete, but what degree of abstractness, what degree of concreteness it implies. The function of the teacher is to present such a species of experience and of knowledge to the pupil as is reasonably well fitted to the degree of physiological maturity of the pupil's brain.

The child's creative power.[1]

In the numerical action and re-action between the child and his environment there are, then, at least two main processes proceeding simultaneously: (*a*) a purely internal development in the child's mind of the abstract idea of any particular number in question (say, five) over which the teacher has only indirect control, and (*b*) an external development by the child of some concrete symbol to

[1] A section contributed by the author to 'Report of a Conference on the Teaching of Arithmetic'; London County Council, 1914. [P. S. King & Son.]

represent that idea, a development over which the teacher has direct control. Further, the concrete external symbol necessarily engages the senses for its recognition. Thus, if this external concrete symbol of the particular number is a spoken word (the name ‘ three ’), the sense of hearing and the muscular or ‘ motor ’ sense are involved ; if it is a group of dots, or (later) the written sign (3), the sight is predominantly involved ; if a handful of marbles, both sight and touch are involved in the normal child, but touch alone in the blind ; in those who are blind as well as deaf and dumb, a tactual symbol alone is possible for the development of number, and as the external symbol of the inner abstract number. It is owing to these last limitations that concrete experience in great variety is so specially necessary in teaching Arithmetic to children defective in one or more senses. Where the teaching, in the case of normal children, is sound, all these means of concrete symbolic representation are employed ; eyes, ears, and hands must all play their parts, and in the early stages often all together.

In still deeper analysis of the actual facts, we find, then, three related processes proceeding simultaneously, now with the emphasis upon one, now upon another :

(*a*) The development of the external concrete symbol itself (a handful of objects, dots, the actual sound of the conventional name (five), the written conventional symbol —arranged here in order of decreasing concreteness, and increasing conventionality, symbolism, and final effectiveness) ;

(*b*) The internal perception by the senses (whether a tactual and visual image of the handful of objects or an image, predominantly visual, of the dots, or a *motor* image of the spoken name, or an auditory image of the spoken name, or an image, predominantly visual, of the written sign) ;

(*c*) The internal and growing abstract idea of the particular number itself.

What precisely it is in the mind that embodies this abstract idea, we do not know ; it is at least a product of the *creative* activity of the mind working, however humbly, in its idealizing and imaginative aspects, through the senses on concrete experience. *This creative element is, throughout*

the most vital in the whole process, and the most characteristic of the child's personality, though the two other elements are indispensable to its birth, growth, and maturity. We may well think of the abstract idea as an internal activity or power of the mind that is ever ready to react when suitably stimulated by experience, and without which experience itself would for ever remain uninterpreted. Now out of all the possible external concrete symbols for the particular number idea, there stand out ultimately two which experience shows to be pre-eminently economical, convenient, and simple for normal pupils. In the long run, as the fittest to survive, these supplant all others ; they are the spoken name (three) and the written figure (3). In what precise forms these symbols appear in memory is not clearly understood. There is increasing evidence to show that auditory, tactual, visual, and motor images all contribute to the mental mastery of each numerical symbol, though in varying degrees. Thus the written and spoken language again prove superior to all other modes of ultimate symbolization.

But the very brevity, simplicity, and yet rich significance of these symbols (the name and the figure), so far removed as they are from the crude, concrete facts supporting them, teach us this vital fact—that their full understanding can only come as the end and not the beginning of a large variety of more concrete and less conventionalized symbols embodying correspondingly varied concrete experiences.

These physical processes, partly conscious, partly and chiefly sub-conscious, are traversed with more or less rapidity with each new number, and also with each new 'sign' (+, −, ×, ÷, &c.), until the final abstract idea of a number in general is reached, symbolized fitly by a letter of the alphabet. But this is, of course, a much later stage of development.

The experience of mature teachers appears to warrant the statement that the above are the main elements of the whole process involved in the realization of number by the pupil. There are doubtless many subordinate elements which the observant teacher will discover for herself.

CHAPTER VII

HOW A FAMOUS ENGINEER TAUGHT HIMSELF ARITHMETIC

The mode of application of Arithmetic to educational purposes appears to me to have been soundly and well exhibited by Bidder, the once famous calculator and engineer, over forty years ago, in a paper read by him, when an old man, before the Institution of Civil Engineers, of which he was President.[1] Yet evidently so little appreciated has it been that, finding myself in substantial accord with his thought, I quote at length from this paper.

I alighted upon it after my own thoughts on the subject had been put in writing; but I judged Bidder's account to be so much the more forcible of the two that I unhesitatingly substitute his words for my own, though his views in certain respects appear to be somewhat extreme.

In truth, the importance of the subject, the force of character and intellect with which this man advocates his principles, the fact that he owed the development of his powers almost entirely to his own unaided efforts, and the inaccessibility of the original paper to the general teacher—

[1] *Minutes of Proceedings of Civil Engineers*, Vol. XV., pp. 251–80, 1856. The whole paper, of which I have quoted about a third, is well worth serious study by all teachers of Arithmetic who can gain access to these *Minutes*. Especially interesting and valuable is the account given of his discovery, '*at a very early age, and when wholly unacquainted with symbolical representation and algebraic expedients*,' of a method for computation, by successive approximation, of Compound Interest, which is equivalent to obtaining a convergent series for the purpose by use of the Binomial Theorem! This method is immensely superior to those at present found in our Arithmetics—and could be taught quite independently of algebra.

all appear to justify amply such amount of quotation from the paper as will serve to indicate clearly the scope of his treatment of the subject.

'I have for many years,' he says, 'entertained a strong conviction that mental arithmetic can be taught, as easily as, if not even with greater facility than, ordinary arithmetic, and that it may be rendered conducive to more useful purposes, than that of teaching by rule ; that it may be taught in such a way as to strengthen the reasoning powers of the youthful mind ; so to enlarge it, as to ennoble it and to render it capable of embracing all knowledge, particularly that appertaining to the exact sciences. . . . It has been urged that, in order to attain eminence in the science of mental arithmetic, there must be an especial turn of mind, an extraordinary power of memory, and great mathematical aptitude. I have endeavoured to examine my own mind, to compare it with that of others, and to discover if such be the case, but I can detect no particular turn of mind, beyond a predilection for figures, which many possess almost in an equal degree with myself. I do not mean to assert, that all minds are alike constituted to succeed in mental computation ; but I do say that, as far as I can judge, there may be as large a number of successful mental calculators as there are who attain eminence in any other branch of learning. As regards memory, I had in boyhood, at school and at college, many opportunities of comparing my powers of memory with those of others, and I am convinced that I do not possess that faculty in any remarkable degree. If, however, I have not any extraordinary amount of memory, I admit that my mind has received a degree of cultivation in dealing with figures in a particular manner, which has induced in it a peculiar power ; I repeat, however, that this power is, I believe, capable of being attained by any one disposed to devote to it the necessary time and attention. In other respects than numbers, I have not an extraordinary memory ; indeed, I have great difficulty in learning anything by rote. I may learn a page of literature, or poetry, but it is no sooner learned than it is forgotten. On the other hand, facts which I have been at some pains to obtain, and which have induced conviction after examination and reasoning upon them, when once fixed in my mind become indelible. In this, however, there is nothing extraordinary,

and the majority of students have experienced the difference between learning lessons by heart and having them impressed on the mind by reasoning, explanation, or experiment. As regards mathematical aptitude, I cannot be mistaken, for when I was associated with large classes, I experienced considerable difficulty in maintaining a fair position, and in no respect have I ever been distinguished for mathematical pursuits; indeed, up to the present time I have no great fondness for mathematical formulae, particularly if they are very abstruse and repulsive in appearance. I fear you may think that I am occupying your time uselessly with these observations, but I feel that I shall base my argument upon a strong position, if I can demonstrate to you, in the outset, that the exercise of mental calculation requires no extraordinary power of memory, and that mental arithmetic can be taught. . . .

'I am not about to lay before you any abbreviated process of calculation; there are no "royal roads" to mental arithmetic. Whoever wishes to achieve proficiency in that, as in any other branch of science, will only succeed by years of labour and of patient application. In short, in the solution of any arithmetical question, however simple or complicated, every mental process must be analogous to that which is indicated in working out algebraical formulae. No one step can be omitted; but all and every one must be taken up one after another, in such consecutive order, that if reduced to paper, the process might appear prolix, complicated, and inexpeditious, although it is actually arranged with a view of affording relief to the memory. And here let me say, that the exercise of the memory is the only real strain on the mind, and which limits the extent to which mental calculation may be carried. It may be imagined that this is somewhat inconsistent with my previous observation that I possess no extraordinary power of memory. But it must be borne in mind that my memory is the limit by which my mental powers are restricted; and that the processes I pursue are all adopted, simply with a view of relieving the registering powers of the mind, i.e. memory. Now, taking you back to your early infancy, endeavour to recall the first things to which your attention was invited. As infants you were first taught to speak;—you were then taught letters;—then the combination of letters into

words; then of words into sentences; and after that you gradually acquired an extensive vocabulary of words and facts. We possess and store these words and facts in our minds, to be occasionally called forth as we need them. For instance, in reading the page of a book, it is clear to me, however rapidly you may read it, that every letter of that page passes in review through the mind.[1] The mind first combines the letters upon the page into words, then the words into sentences, and, from those sentences, it extracts the meaning. Now, in mental calculation I have accumulated, not a very great number of facts, after all;—but I do possess them, and although at this moment I am unconscious of their being so stored up, yet the moment I have a question to solve I have them instantly at command. And it appears to me that, in both cases, the phenomena may be compared to that which we have all observed in Nature. If, on a dark night, there occurs a storm of lightning, during the instant of the flash, although immeasurable in point of time, every object is rendered clear, and out of that view, so placed before us, we can select some one object for our consideration. So I believe it is in the mind; whenever, as in calculation, I feel called upon to make use of the stores of my mind, they seem to rise with the rapidity of lightning. The reasoning faculty seizes upon a particular series of facts necessary for the purpose, deals with each fact according as the circumstances require, and transmits it to the memory for registration. But the registration required for figures is very different from that demanded for ordinary occurrences. An author, when writing on any subject, first forms the argument in his mind, or frames an outline of the plot which he proposes to fill up; but in the mode of recording his views in writing, he has the advantage of an infinite variety of combinations of words, more or less clear and expressive, without feeling restricted to any particular words, or form of expression. But in mental computation there is no such latitude; if you wish to commit arithmetical processes to paper, you must record them in the exact form in which you have reasoned on them, and in their exact sequence and order; a wrong figure, or a figure misplaced, would vitiate the whole result, and hence the great strain on the mind occasioned by mental computation; everything must be remembered with perfect

[1] Subconsciously *perhaps*: certainly not *consciously*.

accuracy, and when the number of impressions to be retained in the mind is large, the retention of them with sufficient distinctness is a work of great mental labour.[1] Hence it is that where the impressions required are few and simple, they are taken up with great rapidity; but in proportion as the numbers increase, so the registration by the mind becomes more and more difficult, until at last the process becomes as slow as registration upon paper. When that point is arrived at, it is clear that the utility of mental calculation ceases, and the process ought to be carried on upon paper. But up to that point the velocity of the mental process cannot be adequately expressed; the utterance of words cannot equal it; in fact, as compared with the process of speaking, or of writing, it is as the velocity of a message transmitted by telegraph to the speed of an express train. I can perhaps convey to you no stronger view of this subject than by mentioning that, were my powers of registration at all equal to the powers of reasoning, or execution, I should have no difficulty, in an inconceivably short space of time, in composing a voluminous table of logarithms; but the power of registration limits the power of calculation, and as I said before, it is only with great labour and stress of mind that mental calculation can be carried on beyond a certain extent. Now, for instance, suppose that I had to multiply 89 by 73, I should say instantly 6,497; if I read the figures written out before me I could not express a result more correctly, or more rapidly; this facility has, however, tended to deceive me, for I fancied that I possessed a multiplication table up to 100 times 100, and, when in full practice, even beyond that; but I was in error; the fact is that I go through the entire operation of the computation in that short interval of time which it takes me to announce the result to you. I multiply 80 by 70, 80 by 3; 9 by 70, and 9 by 3; which will be the whole of the process as expressed algebraically, and then I add them up in what appears to be merely an instant of time. This is done without labour to the mind; and I can do any quantity of the same sort of calculation without any labour; and can continue it for a long period; but when the number of figures increases, the strain on the mind is augmented in a very rapid ratio. As compared

[1] The danger of overstrain in early arithmetical teaching is still insufficiently recognized.

with the operation on paper, in multiplying 3 figures by 3 figures, you have three lines of 4 figures each, or 12 figures in the process to be added up; in multiplying 6 figures into 6 figures, you have six lines of 7 figures, or 42 figures to be added up. The time, therefore, in registration on paper will be as 12 to 42. But the process in the mind is different. Not only have I that additional number of facts to create, but they must be imprinted on the mind. The impressions to be made are more in number, they are also more varied, and the impression required is so much deeper, that instead of being like 3 or 4 to 1, it is something like 16 to 1. Instead of increasing by the square, I believe it increases by the fourth power. I do not pretend to say that it can be expressed mathematically, but the ratio increases so rapidly that it soon limits the useful effect of mental calculation. As a great effort I have multiplied 12 places of figures by 12 places of figures; but that has required much time, and was a great strain upon the mind. Therefore, in stating my conviction that mental arithmetic could be taught, I would desire it to be understood, that the limits within which it may be usefully and properly applied, should be restricted to multiplying 3 figures by 3 figures. Up to that extent, I believe it may be taught with considerable facility, and will be received by young minds, so disposed, quite as easily as the ordinary rules of arithmetic. The reason for my obtaining the peculiar power of dealing with numbers may be attributed to the fact, that I understood the value of numbers before I knew the symbolical figures. I learned to calculate before I could read, and therefore long before I knew one figure from another. In consequence of this, the numbers have always had a significance and a meaning to me very different to that which figures convey to children in general. If a boy is desired to multiply 3,487 by 3,273 he goes through a certain process, which he has been taught dogmatically; he cannot explain the process, or the reasons for adopting it, but he arrives, almost mechanically, at the amount,—11,412,951; which he has been taught is the result he should obtain, without any appreciation of what the figures represent, or how he arrives at them. The process may, without exaggeration, be compared to the task of committing to memory a page of letters, instead of a page of words. Most of us would, without much

difficulty, undertake to learn by heart a page of either prose or poetry, but there are few among us who would undertake the same task with a page of letters. In fact there would be just the difference between attempting to remember a telegraphic message, transmitted by a code of arbitrary signals, and one sent in plain words.

. . . .

'As nearly as I can recollect, it was at about the age of six years that I was first introduced to the science of figures. My father was a working mason, and my elder brother pursued the same calling. My first and only instructor in figures was that elder brother. The instruction he gave me was commenced by teaching me to count up to 10. Having accomplished this, he induced me to go to 100, and there he stopped. Having acquired a certain knowledge of numbers, by counting up to 100, I amused myself by repeating the process, and found that by stopping at 10, and repeating that every time, I counted up to 100 much quicker than by going straight through the series. I counted up to 10, then to 10 again = 20, 3 times 10 = 30, 4 times 10 = 40, and so on. This may appear to you a simple process, but I attach the utmost importance to it, because it made me perfectly familiar with numbers up to 100; they became, as it were, my friends, and I knew all their relations and acquaintances. You must bear in mind, that at this time I did not know one written, or printed figure from another, and my knowledge of language was so restricted, that I did not know there was such a word as "multiply"; but having acquired the power of counting up to 100 by 10 and by 5, I set about, in my own way, to acquire the multiplication table. This I arrived at by getting peas, or marbles, and at last I obtained a treasure in a small bag of shot: I used to arrange them into squares, of 8 on each side, and then on counting them throughout I found that the whole number amounted to 64: by that process I satisfied my mind, not only as a matter of memory but as a matter of conviction, that 8 times 8 were 64; and that fact once established has remained there undisturbed until this day, and I dare say it will remain so to the end of my days. It was in this way that I acquired the whole multiplication table up to 10 times; beyond which I never went, it was all that I required.' Bidder then relates a little anecdote of the way in which others

discovered his powers of calculation, in his correct replies to questions concerning the multiplication of single figures, such as 9 times 9, &c. He proceeds :—

'They then went on to ask me up to two places of figures ; 13 times 17 for instance ; that was rather beyond me, at the time, but I had been accustomed to reason on figures, and I said 13 times 17 means 10 times 10 plus 10 times 7, plus 10 times 3 and 3 times 7. I said 10 times 10 are 100, 10 times 7 are 70, 10 times 3 are 30, and 3 times 7 are 21 ; which added together give the result, 221 ; of course I did not do it then as rapidly as afterwards, but I gave the answer correctly, as was verified by the old gentleman's nephew, who began chalking it up to see if I was right. . . .

. . . .

'Then of course my powers of numeration had to be increased, and it was explained to me that 10 hundreds meant a thousand. Numeration beyond that point is very simple in its features ; 1,000 rapidly gets up to 10,000 and 20,000, as it is simply 10, or 20 repeated over again, with thousands at the end, instead of nothing. So by degrees, I became familiar with the numeration table up to a million. From two places of figures, I got to 3 places ;—then to 4 places of figures, which took me up of course to tens of millions ; then I ventured to 5 and 6 places of figures, which I could eventually treat with great facility, and as already mentioned, on one occasion I went through the task of multiplying 12 places of figures by 12 figures ; but it was a great and distressing effort Now, gentlemen, I wish particularly to impress upon you, that in order to multiply up to 3 places of figures by 3 figures, the number of facts I had to store in my mind was less than what was requisite for the acquisition of the common multiplication table up to 12 times 12. For the latter it is necessary to retain 72 facts ; whereas my multiplication, up to 10 times 10, required only 50 facts. Then I had only to recollect, in addition, the permutations among the numbers up to a million, that is to say, I had to recollect that 100 times 100 were 10,000, 10 times 10,000 were 100,000, and that ten hundred thousand made a million. In order to do that I had only the permutations on 6 facts, which amounted to only 18 in number, therefore all the machinery requisite to multiply up to 3 places of figures was restricted to 68 facts ; whilst the ordinary multiplication table,

reaching to 12 times 12, required 72 facts.' [The attention of teachers is particularly drawn to Bidder's remarkable multiplication-table.] 'Now, the importance of this is not perhaps immediately apparent to you, but let me put an example to you. If you ask a boy abruptly, "What is 900 times 80'; he hesitates and cannot answer; because the permutations are not apparent to him; but if he had the required facts as much at his command as he had any fact in the ordinary multiplication table, viz. that 10 times 10 = 100, and that 900 times 80 was nothing more than 9 times 8 by 100 times 10, he would answer off hand 72,000; and if he could answer that, he would easily say 900 times 800 = 720,000. If the facts were stored away in his mind, so as to be available at the instant, he would give the answer without hesitation. If a boy had that power at his command, he might at once, with an ordinary memory, proceed to compute and calculate 3 places of figures; but then there is an essential difference in the mode of manipulation, adopted by the mind, and when recording it on paper. On paper when you multiply any number of figures you begin with the units' places and proceed successively to the left hand, and then you add them up. That process is impracticable in the mind; I could neither remember the figures, nor could I, unless by a great effort, on a particular occasion, recollect a series of lines of figures; but in mental arithmetic you begin at the left hand extremity, and you conclude at the unit, allowing only one fact to be impressed on the mind at a time. You modify that fact every instant as the process goes on; but still the object is to have one fact, and one fact only, stored away at any one time. Probably I had better commence with an instance or two: there are (pointing to the board) 373 by 279; I mark those two numbers down at haphazard, the result of that is 104,067; now the way I arrive at this result is this—I multiply 200 into 300 = 60,000, then multiplying 200 into 70 gives 14,000, I then add them together, and obliterating the previous figures from my mind, carry forward 74,000; I multiply 3 by 200 = 600, and I add that on and carry forward 74,600. I then multiply 300 by 70 = 21,000, which added to 74,600 the previous result, gives 95,600, and I obliterate the first. Then multiplying 70 by 70 = 4,900 and adding that amount, gives 100,500. Then multiply 70 by 3 = 210, and adding as

before, 100,710. I then have to multiply 9 into 300 = 2,700, and pursuing the same process brings the result to 103,410; then multiplying 9 into 70 = 630, and adding again = 104,040; then multiplying 9 into 3 = 27, and adding as before, gives the product 104,067. That is the process I go through in my mind. "Take another example; for instance, multiplying 173 × 397 the following process is performed mentally:—

100 × 397 = 39,700							
70 × 300 = 21,000; = 60,700							
70 × 90	= 6,300; = 67,000						
70 × 7		= 490; = 67,490					
3 × 300			= 900; = 68,390				
3 × 90				270; = 68,660			
3 × 7					= 21; = 68,681		

The last result in each operation being alone registered by the memory, all the previous results being consecutively obliterated until a total product is obtained.

To show the aptitude of the mind by practice, the above process might be much abbreviated, for I should know at a glance, that

400 × 173 = 69,200
and then . . 3 × 173 = 519
the difference being . . 68,681 as above."

'Now, gentlemen, it must be apparent, and must be received as an established fact, that reduced to paper, mental processes do not recommend themselves as expeditious; but, that, on the contrary, they are often very prolix; they are in reality, solely designed to facilitate the registration in the mind. Although I did that sum almost instantaneously, every one of those processes was performed in my mind; but when, as you saw, I had to register them on the board, that process could not be recommended as either short, clear, or satisfactory. You will see, however, that the process I adopt is, as it were, a process of natural algebra. I have, in fact, worked out this algebraic formula $[a+b+c]\times[d+e+f]=ad+ae+af+bd+be+bf+cd+ce+cf$.

'*Fortunately for me I began by dealing with natural instead of artificial algebra. No man can carry any number of unmeaning symbols in his mind, but I had to deal with numbers which I understood, and I believe it was because my tuition began with this natural mode, that I attained the power I now*

possess;[1] and I think it will be apparent, that teaching arithmetic in this manner, is that which is most likely to recommend itself to beginners, because you are enabled to show them, at every step, that the operation which they are called upon to execute, is that which is right in itself, and will satisfy their reason; and it has this further advantage, that unlike the ordinary mode of teaching arithmetic, which is by dogmas, the mind, instructed in the way I recommend, would have its reasoning powers generally strengthened; *it would be taught to rely on itself*,[2] and thus one of the great objects of education,—that of strengthening the reasoning powers and the resources of the mind—would be generally promoted. I now propose . . . to show you how, step by step, I proceeded to apply the same process to other rules, even up to the extraction of square and cube roots,—compound interest, and the investigation of prime numbers. For all these questions it was necessary to invent my own rules, as I received no suggestion from any one, to assist me. All that was ever explained to me was the meaning of the square, or the cube root, or whatever was the particular branch of arithmetic to which my attention was directed; for as I said before, to show to what a limited extent my education had advanced, when I commenced seriously to calculate, my vocabulary was so restricted, that I did not even know the meaning of the word "multiply". The first time I was asked to "multiply" some small affair, say 23 by 27, I did not know what was meant;—and it was not until I was told that it meant 23 times 27 that I could comprehend the term; I believe, however, that it is not unimportant, that I should have begun without knowing the meaning of the conventional term "multiply", because the words 23 times 27 had to my comprehension a distinct meaning; which was—that 23 times 27 meant 20 times 20, plus 20 times 7, and 3 times 7 plus 20 times 3. It must be evident then, that the powers I possess are derived from careful training; which resulted very much from accident at first, and I think this want of knowledge of terms was one of the accidents, that particularly favoured my progress in arithmetic. . .

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[1] The italics are mine, not Bidder's.
[2] The italics are mine.

'With the exception of Bonnycastle, I do not know, that I ever opened a treatise on arithmetic in my life. . .

. . . .

'My object has been especially to call your attention to the fact, that mental calculation depends on two faculties of the mind, in simultaneous operation,—computing and registering the result;—the faculty of computing depending on the mind having a store of facts at its command, which it may summon to its use, without apparent effort; and the latter,—the registering,—depending on the tendency of the processes to bring all calculations, as far as it may be practicable, into one result, and to have that one result alone, at a time, registered upon the mind. *I have laid great stress on the importance of beginning to study numbers and quantities naturally, before being introduced to them through the medium of symbols*;[1] and I am confirmed in this opinion, by the fact that already, in consequence of my remarks, several gentlemen who have applied them practically, acknowledge that the chief difficulty they have experienced, has been in retaining in their memory *the figures representing the numbers with which they proposed* to deal, and not the *numbers themselves*.[1] I believe that much of the facility of mental calculation, and also of mastery over numbers, depends on having the idea of numbers impressed upon the mind, without any reference to symbols, The number 763 is represented symbolically by three figures 7, 6, 3; but 763 is only one quantity,—one number,—one idea, and it presents itself to my mind just as the word 'hippopotamus' presents the idea of one animal. Now if you were called upon to represent the animal 'hippopotamus' by the figures 174754, it would be far more difficult to remember, because those figures have no relation to one another—they do not guide to another sequence; and hence I feel—and it is an opinion, on which, the more I reflect, the more I am confirmed, that you should have numbers impressed on your mind as an idea connected or identified, with themselves, and not through the 'dry bones' of figures. The word 'mind', if recollected merely in connexion with four symbols, or the four letters M I N D, would create a much greater difficulty to the memory, than the word 'mind', with which a signification is immediately

[1] The italics are mine.

associated. I have already pointed out to you, that, within certain limits, the power of registration keeps pace with computation ; but that when such limit was passed, mental computation could no longer be used with advantage. I have fixed that limit at multiplying 3 figures by 3 figures ; and I do not assign that limit without reason. Each set, or series of 3 figures, constitutes a step in numbers, 787 is one series,—the second series is 787 thousand, the next series 787 millions, the next 787 thousand millions, and the next 787 billions. Therefore, at the change beyond each third figure, another idea must be seized by the mind ; and though it is but one idea, yet with all the training I have had, when I pass three figures, and jump from 787 to 1,787, I cannot realize to myself that it is but one idea ;—in fact there are two, and this increases the strain on the registering powers of the mind. In explaining the process of multiplication I pointed out to you the necessity of keeping only one result before the mind at a time ; and you will find, throughout the whole of the remarks I shall have to submit to you, that the same plan is pursued, and that, wherever it is practicable, one result alone is presented to the memory for registration. I must impress upon you that this is the key to all the other processes in arithmetic. Whoever is master of the multiplication table, and will make it his own in the way I have described, will be at no loss to find for himself a method of applying it to every other branch of arithmetic. In dealing with figures, it confers the same kind of advantage over a person who only knows numbers through symbols, as would be possessed by a man judging of the general contour of a country from an eminence as compared with the observations of a man attempting to view it from between two hedges.'

. . . .

'With respect to Addition and Subtraction I have little to observe, because I follow the same system as in multiplication ; beginning with the left-hand figures and proceeding consecutively to the right. By this means I have only one result to register ; as I get rid of the first series of figures I have no necessity for keeping in view the numbers with which I have to deal. It does not follow that I do not recollect them ; on the contrary, I invariably bear them in mind ; but my object is always to relieve the mind from the feeling of oppression arising from the necessity of keeping an accurate

record, and to seek for that relief by dealing with the other parts of the operation, in such manner as to accomplish it; for the only strain I have experienced, has been whenever the registering power is at all oppressed.' . . .

. . . .

'It only remains for me to lay before you the mode by which, I think, mental arithmetic should be taught;' . . .

'*I think it most essential that numbers should be taught before figures*—that is to say—*before their symbols and probably even before the letters of the alphabet are learned.*[1] The first step should be to teach the child to count up to 10 and then to 100. He should then be instructed to form his own multiplication table, by connecting rectangular pieces of wood, shot, or marbles, or any symmetrical figures: probably marbles may be the best, as they are the very early associates of the child, and may be considered in some degree as his playmates, and will therefore be likely to form the most agreeable associations in his mind. Having formed these rectangles, he will be enabled by his previous experience in counting, to reckon the number of pieces in any rectangle, and thus to demonstrate to himself all the facts of the multiplication table, up to 10 times 10. Having thus acquired the multiplication table, up to 100, he should then be taught to count up to a 1,000 by 10's and 100's. It would not then be difficult to teach him to enlarge his own multiplication table. In the first place, he would have no difficulty in multiplying 10 by 17, because he will be quite familiar with the fact that 10 times 10 are 100, and 10 times 7 are 70, and adding them together will give the result, represented by 170. It will then be easy to follow this by multiplying 17 by 13. He knows already that 10 times 17 are 170 and that 3 times 10 are 30, which added gives 200, and that 3 times 7 are 21, which added gives 221, the result required. By patience and constant practice in this way, he would gradually be taught to multiply 2 figures by 2, and eventually 3 figures by 3.[2] After this he will be led upon the same principles to the application of his faculties to the other rules of arithmetic.

'But I would suggest that this mode of proceeding presents

[1] The italics are mine.

[2] All, observe, without the use of figures to represent numbers.

advantages of much greater importance than even the teaching of figures; for far beyond the mere facilities in computation, would be the advantages afforded by the opportunity of making this branch of education conducive to the highest objects to which education can be directed; that is, to the cultivation of the reasoning powers in general. I would, therefore, introduce a boy, through this means, to natural geometry and algebra. By placing shots, or any small symmetrical objects on the circumference and the diameter of a circle, he would be able, by actual observation, to satisfy himself of their relative proportions. He might simultaneously be taught the relation of the area of the circle to the area of the square. He might also be taught the beautiful problem that the square of the hypothenuse equals the squares on the other sides of a right-angled triangle—that the areas of all triangles on the same, or equal bases, and between two parallel lines are equal. Of these, and many other useful facts, he would satisfy himself, long before he could appreciate the methods by which they are demonstrated in the elementary works on mathematics.

'Advantage may also be taken of this mode to develop many other ideas connected with geometry, as, for instance, that all the angles subtended from the same chord in the circle are equal. This might be shown by having a small angle cut in pasteboard, and fitted to every possible position in which two lines could be drawn within the circle upon the same chord. He might also be taught that the rectangles of the portions of any two lines intersecting in a circle are equal... So again as regards the series $1+3+5+7+9$ (&c.): the summation of this series is equal to the square of the number of terms required to be summed up. If the learner once acquired a feeling for the beauty of the properties of figures—surmising that he had any natural taste for arithmetic—the discovery of these facts by his own efforts might incite him to farther investigations, and enable him to trace out his own path in the sciences. I would again, however, observe, that I should despair of any great success in the pupil's progress in the science of arithmetic if he did not commence before he knew anything of symbols, and if his first conceptions of numbers were not derived from their real tangible quantity and significance.'

The principles embodied in these extracts may be epitomized as follows :

1. Arithmetical facts should be gained concretely by direct appeal to the senses.

2. This is to be effected by the pupil himself in the arrangement of groups of marbles or shot, &c.

3. The aim throughout is to develop the resources of the mind, and to encourage the growth of self-reliance.

4. It appears that the way to success in practical calculation is only to be reached indirectly by thorough familiarity with arithmetical truths gained by the pupil's own reason, observation, and inventiveness.

5. The dogmatic forcing of abstract rules is wholly to be avoided. Self-reliance and inventiveness are to be encouraged to the utmost. 'It is impossible to exaggerate the importance of this feeling of confidence,' says Thring, speaking of the growth in the pupil of the feeling of confidence in the correctness of his results, due to his own clear and right thinking—apart from any assurance from master or from book.

6. Written arithmetic (i.e. special symbols for numbers and operations with numbers) is not to be introduced until the pupil's mind sees the necessity for it in the difficulty of registering more than a certain amount in the memory.[1]

This limit, Bidder considers, is reached when 'the pupil can multiply' together readily two numbers consisting

[1] To judge by many of the books expository of Kindergarten Teaching, one observes that the above principle is neglected here also as well as in ordinary schools. The written symbols for numbers are introduced, it appears to me, much too early. It is a fact highly significant to the thoughtful that the difficulties of notation (the art of expressing numbers by written symbols) should always have so much exceeded those of numeration (the art of counting by use of spoken symbols). I take the explanation mainly to lie in the fact that insufficient attention is paid to the rate of development of the child-brain. The difficulties alluded to surely indicate that such abstractions (as these written symbols) have been presented prematurely. That which is an almost insuperable difficulty to the youthful mind at one age is frequently assimilated with great ease at a somewhat later age. This is gradually being recognized by experts in the case of ordinary writing: but the recognition and application of the principle throughout our educational system are astonishingly small. 'Festina lente' should be the practical interpretation of the physiological facts I allude to above. At the same time the above principle, rigorously interpreted, would lead to a dangerous monotony in infant teaching. See last paragraph in this chapter.

of three figures each. (It will be observed that this is far beyond the point which traditional methods reach, and probably, within a reasonable period of time, much beyond the capacity of the average child, however well taught. We have to remember that Bidder undoubtedly possessed a special talent for calculation, and therefore overestimates the ordinary capacity. Nevertheless, his experience is of great value, and his opinion worthy of serious attention.) This principle I conceive to be of great importance. It is certainly true that mental arithmetic is often advocated and, to a small extent, taught; but only in conjunction with ordinary or written arithmetic. Its pre-eminent office appears to have been much misunderstood. At present, written arithmetic is the standing dish, while mental arithmetic is a mere sauce, palatable or otherwise—which may therefore be taken or left. Exactly the contrary, it appears to me, should be the case in education: mental arithmetic should come first and form the solid food: written arithmetic should be the luxury, given where and when it can be appreciated. Foreigners are often struck with the meagre capacity for mental arithmetical calculations of even otherwise well-educated English people. How many of us, in taking a railway ticket or making several purchases in a shop, are able to check mentally and rapidly the correctness of the change? Truly this incapacity is partly due to the complex system of our units, but substantially it is the result of deficient school practice in mental calculations.

I am convinced that our haste to introduce written symbols of whatsoever nature in the education of the young, gives rise to hosts of artificial difficulties at the very outset which produce mystification and servility of intellect in the pupil, and dogmatism and lack of sympathy in the teacher: further, that these symbols, if accepted at a more mature stage of growth of the young brain, would be appreciated at their proper value, and consequently used in their proper place. Difficulties, unavoidably springing from them previously, would now never appear at all. Necessary and inevitable as all abstract symbolism is for the registration of truth, at the same time it frequently assumes the form of a necessary evil. We are all continually having to repent that we misread the symbol

in place of realizing the fact. Especially in the education of the very young is all special symbolism to be deferred to as late a period as possible, and thereafter even to be avoided as far as is feasible, by strenuously accustoming the mind to attain ultimately unto an intuitive perception of the truth that comes, and comes only, from thorough familiarity with the varied aspects of the concrete. 'Symbols are the money of fools, but only the counters of wise men.'

To proceed with the principles inculcated by Bidder:

7. The incomparable superiority of mental over written calculations, up to a certain point (as this would appear to be inclusive of the product of three figures by three, we may safely claim this superiority in all ordinary affairs of life) :—the obtaining one result in the calculation at a time and this result only to be registered in the memory till replaced by another: above all, the attaining of these successive results in the order of their numerical importance, so that, if the process is stopped at any stage we have at all events a more or less close approximation to the truth.[1]

[1] We have an excellent instance of this in the process used by Bidder for calculation of Compound Interest. It is to be noted that his method for Multiplication leads readily to the useful but still insufficiently practised process called 'contracted Multiplication.' I take this opportunity of recommending to the notice of those teachers who are not yet acquainted with them, a little book on *Decimal Approximations*, by H. St. John Hunter (Macmillan, 1892), and a chapter on 'Arithmetical Precision' in *Thirty Years of Teaching*, by L. C. Miall (Macmillan, 1897). The general principles applicable to approximate calculations (of which kind are *all* concrete calculations necessarily) and special methods for their application, are points of the very highest importance, which nevertheless have been largely neglected in the teaching of Arithmetic. The results are seen in the gross blunders made by scholars and others when application of their arithmetical knowledge is made to the concrete sciences. The educational advantages of perfect accuracy in the answer to an arithmetical problem—though incontestably great—have been unduly exaggerated, at the expense of the benefits that would accrue from teaching the mind to judge of the relative importance for the purpose in hand of the various figures employed in the process of solution. It is significant that, over seventy years ago, De Morgan, referring to circulating decimals and the rules given in books of Arithmetic for reducing them to vulgar fractions, remarks—

'We would recommend to the beginner to omit all notions of these fractions as they are of no practical use, and cannot be thoroughly understood without some knowledge of algebra. It is sufficient for the student to know, that he can always either reduce a common fraction to a decimal, or find a decimal near enough to it, for his purpose, though

8. The gradual introduction of Algebraic ideas, as a form of Universal arithmetic, in place of the highly abstract science as still presented to the learner. As Bidder naïvely remarks, 'Fortunately for me I began by dealing with natural instead of artificial algebra'.

9. The introduction to geometrical ideas by measurement of concrete objects and figures.

There is clearly need for systematic experiment in this field of education—the early stages of arithmetical teaching.

the calculation in which he is engaged requires a degree of accuracy which the finest microscope will not appreciate.' (*On the Study and Difficulties of Mathematics*, p. 17, 1836.) While largely agreeing with so experienced a teacher as De Morgan, we advocate some attention to the idea of a circulating decimal as intrinsically worthy of discussion.

CHAPTER VIII

ILLUSTRATION OF TYPES OF PROOF OR EVIDENCE SUITABLE FOR BEGINNERS IN GEOMETRY, AND THE APPROPRIATE ORDER OF THEIR DEVELOPMENT.[1]

THE following brief notes on lessons are inserted mainly for the benefit of young teachers and teachers in training. They may also be suggestive to older teachers desirous of reaching a balance between the methods of the out-and-out experimental school and those of the old Euclidian school. The methods are further illustrated and developed in the immediately succeeding chapters.

The method is suitable, also, in essence, with a more rapid passage to the scientific type of proof, throughout mathematical education.

The central truth here to be presented to our pupils is :

Whenever two straight lines cross, they form two pairs of equal and vertically-opposite angles.

The syllabus should be so arranged that every central truth should arise in response to some demand springing from a practical problem, involving some manual as well as mental activity.[2] Illustrations of this will be given later, when a whole series of truths about the congruency of triangles (equivalent to the substance of Euclid I. 4, 8, 26, and 32) will be evolved in response to stimuli arising out of the practical problem to construct a map of any group of natural objects (i.e. from Geographical correlations).

[1] For further examples see D. Mair's *School Course of Mathematics* (Clarendon Press, Oxford).

[2] Notes of Lectures to Saturday Morning Class of Teachers: Sunderland, 1903–4.

No special practical problem is, in this case, mentioned, as central truths may arise from a variety of practical problems according to the class and teacher.

The main point is that an emotional need be felt for the truth, and thereby interest in its development awakened.

The procedure is to be heuristic throughout.

[Only brief outlines are described ; the number of lessons required will depend on circumstances.]

Numerical applications and illustrations, generally taken from the surrounding world, are to be copious throughout.

Previous knowledge assumed :—

(1) Meaning and measurement of angles.

(2) Two adjacent angles equal together 180°, or two right angles.

[This truth, (2), itself, and those implied in (1) should have been discovered and assimilated by a procedure similar in essence to that now to be described.]

The meaning of the new term (vertically-opposite) now to be used, is to be impressed by copious preliminary examples. (For type see examples at end.)

We have now to consider the way in which the truth in question and the cumulative evidence for it can be most suitably presented to the pupil.

The successive stages or types of evidence or proof appear to be three in number :—

A. Experimental.
B. Intuitional.
C. Scientific.

A. Experimental Evidence. This kind of evidence gives the mind, through the senses, the concrete material for knowledge, and suggests general truths, also accustoms pupils to new terms, e.g. vertically-opposite.

Thus, draw any two intersecting straight lines.

(1) Pupils are to estimate by eye the size (in degrees) of each of the four angles thus obviously formed, and to arrange them in order of magnitude.

Result :—P appears $= Q$.
R appears $= S$.

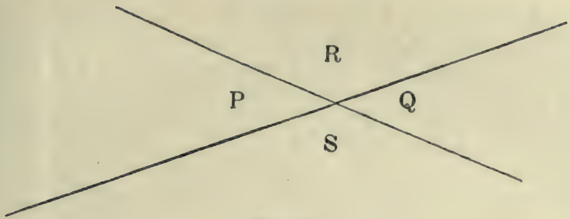


Fig. 21.

With practice, children in general will estimate correctly to within a few degrees (sometimes one or two). In actual practice, $PQRS$ above will be each a number of definite degrees. Note when children fail to obtain $P+R = 180°$, $R+Q = 180°$, &c.

This series of lessons is not of course to be slavishly imitated, but simply the spirit noted. Often the actual answers received will cause quite a different order of detail to be appropriate, but it will be found that when the spirit of this natural[1] method has become habitually present to the teacher's mind, the essence or principle of the procedure will become unconsciously adopted as most appropriate to the pupil's understanding, though the details in its application will and should differ with different classes and teachers. The teacher who, by force of reflection, has reached the style of habitual unconscious application of sound principles has become a veritable *artist* in his work. But every new class, even every new child, is a new problem to the earnest, thoughtful young teacher, and even well-grounded principles should be constantly re-tested and applied in new lights.

More exact tests :—

(2) Cut out a paper angle $= P$ and superpose on Q, &c.

(3) Also measure P, Q, R, S, with protractors.

(4) Actually cut out P and superpose on Q, &c. (each pupil having previously drawn any pair of intersecting straight lines so that a great variety of cases are actually tested simultaneously. This produces greater conviction).

Result: State the discovery in your own words (note the

[1] *Natural*—because the method coincides essentially with the process by which the race itself discovered and established such truths.

training in literary and precise expression here, and also in drawing a conclusion suitable to the evidence offered) :—e.g. *as accurately as we can measure, in all the different cases tried, the vertically-opposite angles are equal.* Write down in notebooks. Note the two parts of this conclusion.

N.B.—Vary the lengths of lines and size of angles.

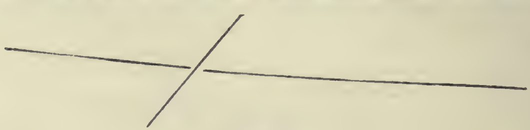


FIG. 22.

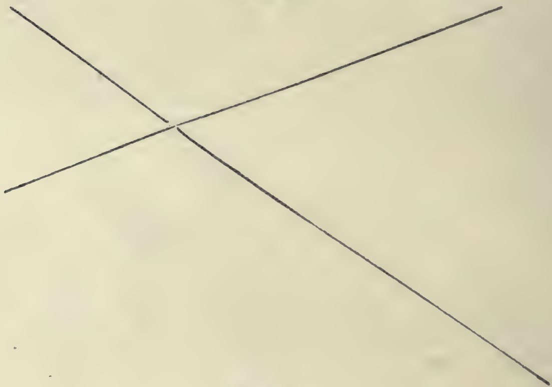


FIG. 23.

FIG. 24.

Clearly all this experimental evidence suggests a certain truth which would be (1) universal or general, and (2) perfectly accurate (i.e. reach perfect equality), if the lines were perfectly straight. [Note these two characteristics of the actual general truth itself, which must be finally proved to the satisfaction of your pupils.]

Note that the whole procedure so far has been one of measurement in actual concrete units: experimental: it has established particular truth approximately but not universal truth perfectly or absolutely. Give also applications to surrounding objects (e.g. window, leaf on stalk, branch on tree, tree on ground, ladder and rungs, scissors, tram crossings, &c., for estimation of angles, &c.).

Gradually develop to clearness the fundamental fact that all actual concrete measurement has perfect accuracy for its ideal but can never reach it.

Many teachers and textbook writers, in neglect of this and other truths, use, with false logic, experimental measurements for the sole evidence of general and accurate truths. We now reach the second stage of evidence—

B. Intuitional Proof or Evidence. This kind of evidence establishes general and accurate truths, but appeals implicitly to postulates of sense-experience whenever necessary: founds the truth on an independent basis of its own by direct appeal to first principles, and not merely as a link in a systematic chain of argument where the emotional strength of the connexion is weakened by the number of previously established truths forming the links of the chain: this latter, systematic kind of proof, the more systematic it is the more it approaches the ideal of scientific proof, and the more scientific it is, the less it is comprehensible by immature minds.

The pupil is now ripe for these questions:—

Is the truth above suggested true in all cases (i.e. generally or universally true)?

Is it perfectly or accurately true?

Ask for general proofs (to which the pupil has been habituated in his previous arithmetical studies, and, to some extent, in his geometrical), and note the kinds of answers attempted and given.

Here take two intersecting rods pinned together smoothly at B—

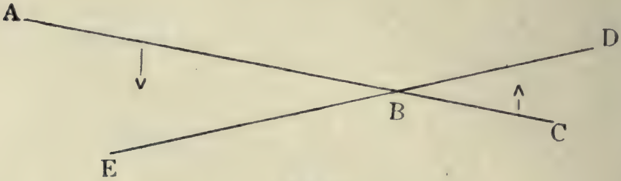


FIG. 25.

(5) Lay AC on DE so that they coincide in direction (not necessarily in length). Children take long before they feel convinced that angles are independent of lengths of arms,[1] showing that the conception of an angle is still very imperfect. A perfect conception, though, observe, is never reached even by the professional mathematician, for the properties connoted by 'angle' are continually being increased by the discoveries of science.

Then slowly turn (or rather let the pupil come up in front of class and turn) AB round B to trace out angle ABE. Then, simultaneously and automatically, we find BC turns round B in the opposite direction, and measures out, of course, the angle CBD. Therefore, amounts rotated being the same, the angles formed are equal.

Repeat the experiment with angles ABD and CBE.

Note the fundamental principles implicitly used in this 'intuitional' proof (i.e. the postulates implicitly appealed to: it may not be advisable to draw the attention of the pupils to them, i.e. to make them explicit), viz. the axiom or postulate that when one part of the straight line rotates so must the other, and, further, that the amounts rotated are equal, &c. Most pupils will quietly and sub-consciously take this for granted on seeing it done, i.e. without being aware consciously of the assumption made. It is precisely in the use of concrete sub-conscious experience of this kind that intuition consists. In contrast, where the sub-conscious experience is dragged out into consciousness and

[1] Probably the difficulty here partly lies in the conception of the 'infinite' involved in the truth (cf. Chap. IV, § 15).

stated as postulate or axiom, then the proof becomes more scientific, but this is a process difficult for children, and confusing, and therefore to be used with caution, and only in gradually increasing doses.

Note the combined use of the muscular, tactual, and visual senses in these experiments.

(5A) Reverse the above experiment. Taking the rods crossed as they are, rotate until they coincide.

Note the use in these two experiments (intuitionally employed for proof) of the rotation or turning measure of angles. The direction measure is employed subsequently (7). Both aspects are important.

(6) *Imagine* the foregoing experiments made.

The pupils are to describe them in their own words, and finally write them in their notebooks. [Note the decreasing appeal to sense-perception here and throughout: the final aim is gradually to substitute reason and imagination for sense-perception.]

(7) Another proof (intuitional also):—

Size of angle depends on direction of lines bounding it: same directions, therefore same or equal angles.

The teacher of course employs, first, an actual figure here, and makes the pupil share in the discovery and statement of the proof as much as possible. Finally, this proof should be stated in general terms, independent of any particular figure.

The teacher may interest himself in finding what sense-perception postulates are assumed in the above seventh piece of evidence or proof.

Note the cumulative effect of these proofs on the emotional conviction of your pupil.

Examples on and practical applications of the above. We now reach the third and final stage of the evidence:—

C. Scientific Proof or Evidence:—

(i.e. dependent more directly on previously proved truths, so as to form gradually a more systematic connexion.)

Perfect scientific proof employs no new sense-perception postulates or axioms, but puts all such at the beginning as the assumed foundation, and thereafter employs nothing

but purely logical reasoning. Science is a minimal problem, but never reaches this, its ideal.

Pupils should be accustomed to discovering these proofs (as others) for themselves, helped and guided by the teacher when advisable. (The different abilities of teachers will show themselves particularly here in the varying success with which they can draw the proof from the pupil, and thus develop his or her originality, self-reliance, and initiative.)

(8) Particular case (to suggest the general scientific proof) :—(See Fig. 21.)

P is measured and found to be, say, 30°. What is R? (calculate without measurement.)

R is $180° - 30° = 150°$, because $P+R = 180°$. What then is Q? Q is $180° - R = 180° - 150° = 30°$. Conclusion? $P = 30° = Q$.

Will this always be so? Can any pupil give me a general proof now, applicable to all kinds of crossing lines?

[Essence of general proof is there, if mind can only see it.] Repeat with other numerical cases if necessary.

(9) Final and general form of (scientific) proof (here, much contracted, of course: the teacher will emphasize the dependence on previous truths) :—

(*i*) $P + R = 180°$ (previous truth).
$R + Q = 180°$ do.
$\therefore P + R = R+Q. \quad \therefore P = Q.$

Prove similarly $R = S$.

(*ii*) The pupils are now to state the above symbolic proof in their own language.

[An excellent and disciplinary exercise this.]

This translation of symbolic into general language is especially necessary in mathematics, more particularly in Algebra, e.g. $(a+b)(a-b) = a^2-b^2$, &c., thus :—

If any two straight lines intersect, then any pair of adjacent angles thus formed equal together two right angles. Now choose any other pair of adjacent angles. It will be seen that to these two pairs *one angle is always common*. Consequently removing this common angle from the equality formed by the two pairs, there remain equal two angles which are seen to be *vertically opposite*.

Some teachers may prefer the general statement first and the symbolic and condensed subsequently.

The teacher will observe—it need not be pointed out to the pupil at this stage—that, though the above is the essence of the so-called rigorously scientific proof in Euclid of this truth or proposition (viz. Book I, 15), it makes implicit use of two additional sense-perception postulates not stated by Euclid in his list of preliminary postulates and axioms, but quietly assumed by him. What are these? The words italicized above give them, viz. postulate I:—The figure is seen when actually drawn to be of such a nature that two pairs of adjacent angles have one angle in common, and, postulate II, on removing this angle the two remaining ones are seen to be vertically opposite. It would, perhaps, be possible to throw these two mutually dependent postulates into one. Now for these postulates we have to appeal to sight, or, in the case of the blind, to touch (i.e. sense-perception). Of course the intuitional proofs implicitly involve these postulates also, and others in addition, and for this last reason are less scientific as involving more assumptions, but easier, therefore, to grasp. As a matter of fact the whole of so-called rigorous scientific proof is riddled with assumptions of this nature, and discovery would be impossible without them as they are constantly exhibiting the actual properties of spacial figures when presented to the senses. Any mathematical science tied down to a limited set of postulates at the start (however numerous), would be inherently incapable of further progress, as scientific progress consists essentially in making conscious the postulates continually assumed and presented in new sense-experience. Finally, then, we see that the ideal of scientific proof is never attainable; consequently (for this and other reasons) we draw the pedagogical conclusion that the rigour or degree of scientific evidence presented must be appropriate to the maturity of the pupil.

Then should come numerous examples and applications. Specimens follow:

(1) Here are two rods. Form a pair of vertically opposite angles with them.

(2) Point out such in the room, in the playground, fields, at home, &c.

(3) An angle is 30°, 50°, 75°. How big is the adjacent angle?

(4) Of two adjacent angles, one is 2, 3, 4, . . . times as big as the other. Construct them, as accurately as possible.

(5) Can you give a general rule for solution of (4) ?

[$P = nQ$, $P+Q = 180°$. Solve and state solution in general language. Or, more simply, $P = 2Q$, $P+Q = 180°$; &c.]

(6) Of two vertically-opposite angles, each is 60°. How big is the other pair ?

(7) Two tram-lines cross. One angle is 32°. Find the other three.

(8) Draw adjacent angles of which one is 30° : 40° : 50°.

(9) Test the above theorem by actual drawing (in a variety of positions of crossing lines).

(10) Construct two crossing lines in which one angle is 30° and another is 50°. Why is this impossible ?

The teacher should gradually make clear the difference in nature between the merely approximate evidence for particular cases or truths established by measurement (as in proofs A. 1, 2, 3, 4, above, under Experimental) and the universal and accurate proofs established by the intuitional and scientific evidence (viz. B. 5, 6, 7, and C. 9).

Moreover, this implies that, in actual measurement, all lines being somewhat imperfectly drawn, the instruments being always more or less defective, the eyes more or less inaccurate in judgement, &c., measurement can never be more than approximate (though none the less practically valuable for that: yet the aim is to reach as great accuracy as possible). On the other hand, the truths established by the other kinds of proof (the scientific, and the intuitional which is merely a preliminary and less rigid species of ideal scientific proof : there is really no sharp dividing line between these kinds of proofs, but only a difference in degree of logical rigour) are universal (so far as we can judge; by the way—experience might suggest exceptions !) and absolutely accurate, only on condition that our lines are absolutely straight, &c. They are therefore ideal truths, but applicable with greater and greater approximation to the actual world in the degree to which we can make our figures perfect, &c.

Finally note that the figure formed by a pair of inter-

secting straight lines is determined in shape by *one* measurement or datum (e.g. any one of the angles formed).

In scientific language, a pair of intersecting straight lines is a one-fold manifoldness.

The main aim throughout is (1) heuristic, affecting the character of the pupil; (2) the gradual substitution of imagination and reason for past sense-perception, as more powerful and economic of mental effort, but consistently with the view of using these as tools for the understanding and discovery of more complex and fresh sense-presented phenomena, i.e. the use of imagination and reason in the interpretation of the ever-growing sensuous world around the pupil. But, while following in essence the principle of parallelism between individual and racial development of experience, let the teacher beware of cutting the ancestral process too short (resulting in mere rote memory work, with no sound basis of sense-experience), and, at the other extreme, of spinning it out too long (resulting in undigested, non-rationalized masses of mere sense-experience). Only experience and reflection can develop in the teacher the art of choosing wisely the happy medium between this Scylla and this Charybdis of mathematical education.

CHAPTER IX

SOME EXPERIMENTS IN TEACHING GEOMETRY TO BLIND CHILDREN[1]

Suggestive use of ideas in the Geometry of Position.

THIS chapter consists of the description of a lesson given by the writer to a class of eight blind children in January, 1903, with the kind permission of their teacher, Mr. G. I. Walker, of Sunderland: and of the description of a lesson given by Mr. Walker himself.

The writer was visiting the class to teach himself rather than others; but as some point accidentally arose suggesting the following lesson, he took the opportunity of trying an experiment. The lesson was therefore entirely unprepared, and is not given as in any way a model, but simply as exhibiting the value for education, even in the special case of the blind, of that branch of geometry (almost entirely neglected in Elementary Education) known as Geometry of Position. The root of its educational applicability lies in the fact that in essence it is the geometrical basis of Geography, or the orientation of natural objects. The Science of Geometry of Position embraces two closely allied aspects: the first dealing with the determination of the position of a body with respect to external bodies, and the second with the determination of the position of the elements or parts of a body relatively to each other. Hence the science embraces the determination not only of simple position but also of shape and size. In future developments these two aspects—so fundamental—of Geometry are reasonably certain to occupy a very central position in Mathematical Education, on account of the varied and numerous correlations that naturally spring from them as the subject

[1] Notes of Lectures to Saturday Morning Class of Teachers, Sunderland, 1903–4.

develops; branching out, as they are capable of doing, into Mensuration, simple Curve-Tracing, and subsequently Analytical Geometry, Practical Plane and Solid Geometry, Trigonometry, Mechanics, Geography, &c., and applicable at different stages of education to Modelling, Art, Manual Work (Wood and Metal), Architecture, Surveying, Dress-making, and Tailoring, &c.

The children varied in age from 9 to 14 years.

The description has been considerably abbreviated, but, as the notes were taken directly after the lesson, the pupils' answers, as here stated, may be relied upon with reasonable confidence.

Important and suggestive for all teachers is the experience of Mr. Walker that, with blind children, the individualities and the standard reached when they came to him for schooling vary so much that—and owing mainly to the former factor—real progress can only be reached by treating them individually as units. Collective class teaching is, broadly speaking, almost impracticable for any reasonable length of time. So far as this experience rests on the varying nature of the individualities, it should appeal with force to all teachers. The neglect of the individualities of our pupils and the consequent failure, broadly speaking, to discover and develop their useful special talents (thus, largely, determining the most appropriate future occupation), and to simultaneously arrest the growth of tendencies prejudicial to themselves and society—all this can justly be charged against much of education.

The practical problem proposed to the class for solution was: How to fix the position of an object (e.g. a book was selected) in space? In other words, if this book I am holding up were removed from its present position, and I wished to be able to replace it in the same position, what observations must I make about its present position? And further, how can I solve the difficulty most simply? Note how many geometrical truths naturally spring from the attempt to solve a practical problem of this kind; in such trials the mind feels vividly the need of geometrical truths, and is irresistibly urged to their discovery and examination. A natural correlation and systematization result. Interest, originality, and a reasonable measure of independence develop. The truth is gradually discovered, examined, veri-

fied, applied, and finally mastered—not necessarily all in one lesson, of course.

N.B.—It is easy to create artificially certain difficulties in criticism of the solution obtained, but the common sense of the teacher guides him as to the degree of subtlety to be introduced.

The book was held up, and after some brief questioning, it was seen that to fix the position of the book as a whole it was sufficient to fix the position of, say, the top cover—a flat oblong surface.

We may now replace the cover by, say, a piece of cardboard of the same size and shape. This again could clearly be replaced by something simpler, viz. the half of it, a triangle obtained by cutting the cardboard along a diagonal, for here when half is fixed so is the whole. The problem now reduces to fixing the position of a simple triangle. (There are, of course, an unlimited number of other ways of dealing with the problem.) To solve this it is advisable to examine the triangle and find out something of its properties. Consequently paper triangles were cut out and given to each pupil to handle—sight, remember, is absent.

Teachers will note the suggestiveness of the lesson with respect to the great function played by tactual and muscular sensations in geometry; this aspect is too much neglected in normal pupils, sight sensations, vastly important as these are, replacing them in too great a degree; all these senses should be developed in mathematical education.

TEACHER: Describe this triangle; tell me something about it.

Pupils feel that it has three sides, also three corners.

TEACHER: Bend it, each of you. Is it still a triangle?

No! Yes! &c. Investigating the 'No' replies, it was agreed that as the bending prevents all the sides from being straight, the bent figure is not a triangle.

N.B.—Much was clearly assumed in this lesson. What was their idea of straightness? Evidently, too, their notion of a triangle more or less obscurely had so developed as to involve straightness of sides.

The idea of flatness or planeness (in the practical sense that a triangle can be applied without bending to the surface of a table) was also felt to belong to their conception of a triangle.

Incidentally it transpired that none of them had any conception of the possibility of producing the sides of a triangle; but this notion, without sight, is obviously complicated unless the figure obtained could be cut out and felt, and even then touch would here be, apparently, vastly more complex to interpret than sight.

Each now placed the triangle flat on the table in front of him.

The original problem was now simplified into this: If each of you take his triangle away and wish to be able to replace it in exactly the same position on the table, what would you do?'

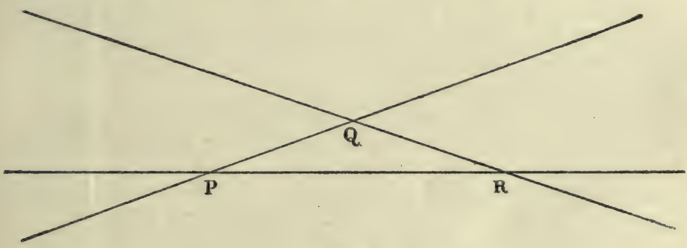


Fig. 26.

Instead of the more complex problem of a triangle in space, it was found necessary to solve first the simpler problem of the position of a triangle in a plane; here the reference framework, instead of being the walls, ceiling, &c., of the room, is the rectangular table itself, a pair of whose edges form an excellent set of rectangular co-ordinate axes—these names were not actually introduced in this lesson, though they, or equivalents, would soon become necessary in further developments.

Boy: 'One could tell if one could see.' Another: 'No! not even then—exactly in the same place, unless one could see very well!' Teacher: 'What then would you do?' Another boy (about 9): 'Make a mark round it as it lies on the table.' Teacher: 'That will do it very well. But, is it necessary to make a mark round all three sides?' Several: 'No!' Teacher: 'How many sides would do?' Boy: 'Two sides.' (The geometrical truth that only a part of

the sides need be marked was not in this instance developed, as a much simpler solution was ultimately reached: in truth, one side only, or two points, suffice—though not for a unique solution.

Developing the idea of position into size, shape, and position, and following the clue given in this answer (for the teacher will note that a chalk-mark round two sides implicitly involves the size of the included angle), another day the teacher could naturally and gradually develop the fundamental truth (corresponding to the substance of Euclid I, 4) that:—A trangle is uniquely determined in shape and size by two sides, and the contained angle, or, equivalently:—If two triangles have two sides of the one respectively equal to two sides of the other, and the angle contained by the two sides of the one equal to the corresponding contained angle in the other, then the triangles are identical in shape and size.

The evidence for its truth could be made, as in Euclid, to rest upon the fact that two such figures can clearly be superposed, and also—still more cogently—upon the actual process of construction of a number of triangles with definite data (e.g. sides 5 in., 6 in., and included angle 32°), when it will be observed that only one kind of triangle can be made with these data. To remove any lingering doubt the triangles made by each member of the class can be all superposed on each other; any triangle not precisely fitting shows up, ultimately, the carelessness of the lad who made it. What modifications would be required for blind children, only the experienced teacher of the blind could, of course, decide. Here suffice it to observe that a good teacher will obtain from an interested class numberless suggestions and clues of this kind, from almost any one of which fundamental geometrical truths can be naturally evolved by the class under the teacher's guidance. In nearly all cases it is important that a great part of the evidence—and especially the preliminary part—should be based upon exact quantitative constructions.

In the present instance the suggestion made by the lad was not developed by the teacher, as it fortunately happened that another lad simultaneously made another suggestion exactly in line with the teacher's own intention at the moment.

Another Boy: 'Three corners would do.' Teacher: 'Yes, if we make three marks at the points where the corners lie, that would do, but are three corners necessary? Will two be sufficient?' Several: 'No!' Teacher: 'Why not?' Several: 'If we knew only where two corners are, we could place the triangle so or so.' See Fig. 27:—If PQ be two points where the corners AB of the triangle ABC are to be placed, the boys showed easily that C might be placed on the table in either of two positions obtained by revolving the triangle round PQ (viz. either at R or S).

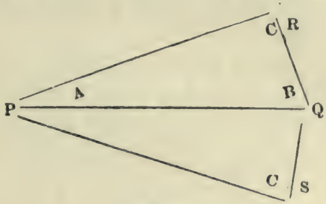


Fig. 27.

In mathematical language we could say in reply to this criticism of the boys that the triangle is determinate in position, but not uniquely; there are only two positions, not an indefinite number as would be the case were only one point given; hence the solution is manifold (or multiple) and determinate—not infinite or as it is termed 'indeterminate'. This language and reply, however, would be unsuitable at the present stage, as its reasonableness rests upon the convention that in deciding determinateness or fixity of position in mathematics, only quantitative data are counted: qualitative data (e.g. right or left, top or bottom, &c.) are often necessary to decide between alternatives, but do not in general affect the mathematical nature of the solution. In the present case no quantitative measurements had yet been actually introduced; they occurred later.

Teacher: 'Then, suppose I have also told you on which side of the two points the triangle lies, how many points or corners would then suffice?' Boy: 'Two.' Teacher: 'Now, will any one state clearly what we have found out about the fixing of our triangle?'

The conclusion reached may be stated thus:—The position of a triangle on the table (i.e. confined to motion in a plane) can be fixed by the aid of two points, provided we know on which side of the line joining the two points it is to lie.

Teacher: 'Now, I want you to tell me this? Mark the two points where the two corners lie; if now you remove the triangle you can put it back in the same position. But, suppose some one came and removed those marks, what precautions could you take beforehand, so as to be able to recover them?'

This is the problem of fixing the position of a point (here, in a plane); a fundamental problem which is inevitably reached in solving any practical problem in orienting bodies. In other words a fundamental problem in geography and geometry is, 'What is my position here and now? How am I to determine or fix it?' Geographically, in maps, the answer leads to the discovery and need of latitudes and longitudes, height above and below sea-level, &c., &c.

Teachers may prefer to start with this simpler form of problem, viz. place a small marble on the table, and ask how it can be replaced if removed, in exactly the same position, thus developing axes and co-ordinates, &c., with the whole practical and yet rationalized paraphernalia of rectangles, parallel lines, parallelograms, &c. It would, in the opinion of the writer, be a great misfortune if one line of development only for mathematical truths in school were to become permanently conventional, thus re-introducing the rigidity of Euclid with all its disadvantages and few of its advantages. It can scarcely be repeated too often that there should be for some time as many different and equally excellent ways of developing geometrical and other mathematical truths as there are efficient teachers, until, in the struggle for existence, a small number of systems survive well-suited, for say a generation, to modern educational needs.

Boy and Girl (replying simultaneously to the question just asked): 'Measure the distance of each corner from the edge of the table' (pointing to the table-edge nearest to themselves).

Teacher: 'That looks hopeful. Let us try it with one corner or point first. Will each of you place one finger-tip on the table? Now, what measurement would you take to get its position?'

Answered by several as before (see above).

Teacher: 'Is this one measurement really sufficient? Suppose the distance is 5 in. Now, from what point of the

table-edge am I to measure to get P (the position of the finger-tip) ?'

BOY: 'Here!' (at Q, foot of perpendicular from P on edge).

TEACHER: 'But, how am I to know where Q is if P is removed ?'

BOY: 'By measuring the distance from Q to corner of table.' (viz. $R\,Q$, see Fig. 28.)

TEACHER: 'How many measurements then must I take altogether to fix the position of one corner of the triangle ?'

BOY: 'Two.'

TEACHER: 'How many measurements then must I make to fix the position of the original triangle ?'

SAME BOY: 'Two for one corner, and one more only for the second corner, for we already know its distance from the first point.'

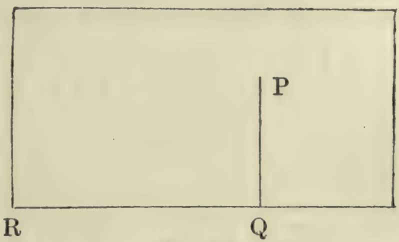


FIG. 28

TEACHER: 'That is altogether— ?'

SAME BOY: 'Three.'

The solution of this lad was very unexpected, and it should be added that it is believed none of the others followed his rapid divination of the simplest kind of solution. To the last question but one we would have naturally expected the answer 'four'—two measurements for each of the two points. And this would have been sufficient, though all four are not necessary. The teacher stopped at this point, as the lesson had been sufficiently long (about 20 minutes or so), but it would have been very interesting to investigate precisely how this lad would have actually worked the problem in a practical numerical case. That he should have quickly jumped to the explicit conclusions:—That two measurements fix, say P′ (one corner), then that P′ P″ (P″ being a second corner of the triangle) being equal to a side (say 3 inches) of the triangle, P'' must lie on a circle, of which P' is centre and radius 3 inches, and therefore that P'' is determined (as one of two positions) by intersection of this

circle with a parallel at proper distance (the third measurement referred to) from edge of table:—all this does not appear in the evidence, but some such equivalent rapid reasoning of an intuitive nature, which, perhaps, he could not explain to others, seems likely. Mr. Walker told the writer that this lad who made so remarkable an answer in this problem of practically fixing the position of an object is very stupid at writing, but very clever at subjects requiring mechanical skill. His age is about 10. This appears to be a distinctly common case among boys, though normal education takes but little account of such inherent and interesting characteristics.

Had it been possible to continue this lesson, the teacher would have followed it up by giving a large number of simple practical problems of a like nature for the pupils to solve (involving, of course, in general, definite measurable units), and the construction of problems whose solution would have required repeated use of the above fundamental principles on the fixing of figures.

The original problem (in space of three dimensions, i.e. ordinary space) might finally have been dealt with.

It should be added that Mr. Walker, the blind teacher of these children, states that one of the chief difficulties in developing the minds of the blind children is that of rousing their interest so as to produce a moral force impelling them to tackle the difficulties that constantly confront them. This deadly lack of interest appears due to the fact that their experience is, relatively to the normal child, almost a blank; thus their minds do not possess that normal stock of ideas and feelings which the teacher can usually successfully appeal to: blind children, when they first come to school, generally and for long live in a species of mental and moral torpor.

In a different degree, of course, but still in a very real sense this same kind of difficulty is often encountered by the teacher with normal children in the teaching of Elementary Mathematics, as much of it is not directed to the experience of the young pupil, so that, relatively to the mathematical teaching, the mind is very similar to that of a blind child, i.e. torpid.

It is fatal to efficiency if the early mathematical teaching (Arithmetic, Elements of Geometry, &c.) fails to take full

and due account of the previous experience of the child (quâ Mathematical) and of its degree of brain maturity.

Lesson by Mr. Walker.

This letter (February 3, 1904) from Mr. Walker contains brief remarks on the lessons just described and a brief description, well worthy of study, of a lesson in geometry which he gave his blind pupils. The portions enclosed in square brackets are added by the writer.

Mr. Walker makes a few brief preliminary remarks to the effect that in his opinion the above description of a lesson (at which he was present) given by the writer is accurate, and then proceeds :—

You say in your notes, that you found that the production of the sides of a triangle was something outside the grasp of the children, and, continuing, you make the very pregnant statement (on p. 106) of the great effort required to grasp the idea of a projected side even when that is shown by a model, which may be touched. This is so to a very striking degree, and I am glad you call attention to it, because it is not generally apprehended. It is sometimes set down to inability of quite another kind, and the study of the question rather discounted as a fruitful one for the mental development of the blind. Rote-memory is too often entirely trusted to, and the consequence is that while a scholar may be able to repeat verbally the different propositions taught him at school, he is altogether unable to bring them to bear upon any matter whatsoever, and is quick to give the whole subject up as one of no use to him. Since your visit I have tried to get my people to see that all solid as well as plane figures may be thought of as being contained within lines. This substitution of lines for sides has helped some of them at least to grasp what is meant by producing the sides of a triangle. The great difficulty in the way of the blind taking hold of mathematical ideas is their inability to picture mentally the figures referred to, and, therefore, they fail to follow descriptions and deductions from them.

[The teacher should try to realize the two cases : (*a*) the normal child taking in almost at a glance the whole outline of a new shape ; (*b*) the painful, difficult, construction of the whole shape, bit by bit, of the blind child compelled to follow the outline with the hand, or to imagine itself following

the outline from the description of the teacher in language necessarily imperfect for the blind because evolved by and adapted to normal people. In answer to Mr. Walker's question as to how she followed the reasoning described below, one little girl said that she imagined she held a small copy of the figure in her hands during the lesson. This suggests the re-introduction of wooden slates with holes and pegs round the sides for the fixing and alteration of strings, in the teaching of geometry to the blind. The blind mathematician, Saunderson (1682–1739) used this device with excellent results. The mathematical feats of this famous man, Lucasian Professor of Mathematics in Cambridge, and blind almost from birth, shows the immense power of touch and muscular sensations in the development of spacial ideas.]

To remedy this I am persuaded the only means is to cultivate the imagination, or power of picture painting. A lesson I gave this morning after I had read your pamphlet will show what I mean, and the means I am taking to get over the difficulty.

I take the wall for a blackboard, and ask the children to follow me as I draw a line. I make a slight noise as I do so, and by questioning find out who is following. (My chief, almost my entire stock of imaginary drawings, are geometrical figures.) I began the lesson by asking them to imagine an equilateral triangle, which I named A B C, reading from the apex. (The base line BC is horizontal.) Then I asked what direction the line BA would take if I produced it. The answer was to the right. What direction will CA take if produced ? Answer, to the left. Now, let BA be produced as far as B is from A, and do likewise with CA. Now, write D and E at the extremities of the produced lines. Suppose now that we join DE, what do you observe ? At once a boy answered the line will be as long as BC. [Note the rapidity and comparative complexity of this intuitive conclusion.] Anything else ? Yes, we have made another triangle. Well, what else do you notice ? Two of its sides are equal. To what ? To two of the sides of the bottom triangle. How many sides of the one are equal to— ? Before I got further a boy answered three. Well, is there anything else ? Then I got the answer I was hoping for. The angle at the apex of the top triangle is equal to the angle at the apex of the bottom one. Let us take these triangles from the wall,

and lay them on the floor. One or two at once said we might lay one on top of the other. If we do, what shall we discover ? PUPILS : The one on top will fit on the one on the floor. TEACHER : We are getting on. Now, let us put the triangles again on the wall, and let us see if we cannot construct some other figure. Suppose we drop a line from *D*, where will it strike ? [Note that, in Mr. Walker's phraseology with the class, to 'drop' a line implies direction vertically downwards, i.e. a plumb-line.] ANSWER : It will strike *B*. Drop one from *E* and tell me where it will strike. ANSWER : *C*. Now, what figure have we ? ANSWER : A four-sided one. Square or oblong ? This evoked much discussion but no unanimity. [The reader is to remember throughout that the pupils are blind, and the imaginary construction of this relatively complicated figure offers great difficulties. A figure is appended at the end, but the reader is asked not to refer to it at present, but to try to realize its form from the verbal description in the text, with the additional advantages over these blind children, that he can easily picture lines and knows the subject matter well.] Well, look from another standpoint. Take *DB* as a base line. After some little time I got the reply : We get another triangle *DAB*. Take *EC* as a base, what do we see ? At once I was told we should get the triangle *EAC*. How many sides of the one equal to certain sides of the other ? or, adding (as if a new thought had struck me), are any of them equal ? I did not need to wait for a reply. It came quick and sharp. Several called out at once that the base *DB* equalled the base *EC*, and that the lines *DA* and *EA* were equal, and the lines *BA* and *CA* were also equal. How many of the one—before I could get out the rest of my question the answer came : three of the one to three of the other. One of the boys added almost immediately the angle at the point *A* in the one is equal to the angle at the point *A* in the other. The same boy followed his remark by this : If the two side angles were equal to the other two it would show that the whole figure was a square. To this remark I simply said, 'Do you think so ?' My reason for so answering was that I did not wish to suggest in any way a line of thought to the lad, who was making splendid efforts to fairly see the bearings of the problem, and I did not think it wise to turn his attention in the slightest from his

trend of thought. This question whether the four-sided figure was a square or an oblong came up earlier in the lesson, and it was evident the lad's mind was working its way to a solution. Now, take the triangle *DAB* out and place it along with the triangle *EAC* on the floor, one on the top of the other. The children at once cried out they are equal. Telling them to put the triangles back to their original positions, I suddenly asked the question : ' How do the sides stand to one another ? ' The answer with very little delay was :—the opposites are equal, the verticals are equal, and the horizontals are equal. I thought I had done enough for one lesson and bade them leave the room. This lesson lasted little over half an hour. I have been working steadily for some time endeavouring to get the children to picture things for themselves. The habit of projecting lines is one which I practise daily less or more in my travels. If I can get an idea as to the position of any place I project a line from some other place near to it which I happen to know, and then follow my guide rope. [The reader must remember that Mr. Walker is himself blind.]

Below is a diagram by Mr. Urquhart, who tested the children by means of a similar diagram in relief, so that he might convince himself that the children really had pictured and accurately followed the verbal description.

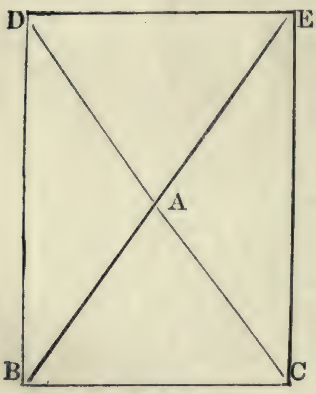


FIG. 29.

The result of this was that the boys tested showed themselves quite familiar with the figure, and pointed and named the different angles as if actually seeing them. [An 'image' or 'figure' to a person born blind is a combination of ideas from touch, muscular and hearing sensations — probably mainly touch. There is, of course, no association of colour sensations with their images of figures. Size and shape have to be constructed without the aid of sight sensations.]

CHAPTER X

SUB-CONSCIOUS EXPERIENCE[1]

The parts played in Education by the sub-conscious and the conscious.

THE experience of teachers of blind children appears fully to harmonize with the experience of accurate observers who have taught mathematics to normal children, respecting the central importance of the educational principles repeatedly emphasized by the writer.

The teacher of the blind, indeed, should be and appears actually to be much more familiar with these educational truths, and with the nature of his pupils, for a misunderstanding of them results in an arrest of development of his pupils' intelligences that is soon brought startlingly into prominence. With normal children corresponding misunderstanding on the part of the teacher also results in arrest of development, but this is obscured by the fact that it is generally not, as is apt to be the case with the blind, an arrest along many lines, but in a more limited number of directions and to a less degree, since so much less of the total environmental stimulus to development is here due to the teacher ; nor is the discovery of the arrest so sharp and rapid as to indicate obviously where the fault lies, being shared perhaps, often, among several teachers. But in each case, sooner or later, any misunderstanding of the natural mode of development of the pupil's mind inevitably produces arrest of development, to a greater or less degree, in one direction or even several. The one real test which ultimately and invariably discovers (often too late) the deficiency in development is the subsequent inability of the pupil to apply the ideas, presumably gained, to the difficulties of some concrete problem whether in or out of school.

The particular educational principle (in addition to the

[1] Notes of Lectures to Saturday Morning Classes of Teachers: Sunderland, 1903–4.

principles expressly mentioned in the text itself) illustrated by and underlying the above lessons is not easily made clear to the young teacher. A peculiar difficulty attaches to any exposition of this principle, because the principle itself must be applied for its own explanation; we are or should be all learners and pupils to the end, and any truth inherently valid for one period of education is, though naturally and in general in a highly different degree, applicable throughout.

The fundamental principle referred to is so vast in its bearings upon all education, so rich in meaning and significance, that only persevering reflection, long continued accurate observation and sympathy with the pupil's stage of mental development can lead the teacher to a reasonable degree of mastery of its content and power to apply it.

Broadly stated, the principle is the intelligent application to education of the following three facts (numbered I, II, III):—

I. Only a relatively small part of the total mental activity rises into the region of consciousness. Consciousness is merely the surface of a vast ocean of sub-conscious mental activity.

[Mental activity here embraces the activity (quâ mental) arising from the functioning not only of the cerebral nervous substance, but of all nervous substance throughout the body.]

Hence the supreme importance of practical exercises demanding the use of eyes, hand, muscle, &c. If we have two eyes, and two ears, we have ten fingers and almost endless muscles. The importance of the hand, even in comparison with the eyes, is evidenced by the use of such phrases as 'He is a good hand at it', applicable to almost any activity. This is not so with 'the eyes', where the application is vastly more limited. Touch is the most fundamental and important sense; were this otherwise the blind and deaf and dumb would be in a sad plight. Students are recommended hereon to read *The Story of My Life*, by Helen Keller (blind, deaf, and dumb).

People with normal senses who are not engaged in manual work are too apt to make 'imagination'—particularly in geometry—coextensive with constructions and memories dependent wholly on the sense of sight, i.e.

imagination is supposed to deal only with sight-images. But, though this is, in consequence, a very common limitation to the use of the word, it is of great importance, particularly for teachers of mathematics, to observe that each sense has its own share in the creation of imagination, so that there is touch-imagination, hearing-imagination, muscle-imagination, &c.

The language of sound (spoken language) appears to have commenced by developing side by side with and by reliance upon the prior gesture-language; and gesture-language to have originated from involuntary reflex-movement. The very words 'image' and 'imagination' are probably derived etymologically from the root of 'imitation'. These facts illustrate the remarks below on III, and bear upon the causes producing many disabilities of the blind in the development of experience and language.

Unfortunately the English language does not appear to possess any single word to denote the function of imagination in general, without regard to the sense from which it may proceed.

This poverty of the language, and our excessive dependence on language in mathematical education (important as language undoubtedly is) are two additional causes of the neglect of the education of the muscular and tactual senses. These senses, as well as sight, are, if appropriately stimulated, vast sources of information and skill which must have their full due in any efficient scheme of mathematical training.

The fingers may share in the thinking as well as the head; the loss of a finger is the loss of so much mental experience and potentiality: whether the finger is actually cut off, or unused (as in much mathematical teaching when drawing, &c., is neglected) is relatively immaterial.

II. Of this conscious part again only a relatively small part (great as that part may absolutely be) is expressible in language.

III. And of this further part expressible in language—a fraction of a fraction—the degree of significance and applicability of the word is (in mathematical education) proportional to the amount of systematized and rationalized sense-impression springing from the external world upon which it is based—which the word in fine represents. The

intimate co-ordination of thought with sense-impressions may be realized from the fact that thought in general passes over into muscular contractions, whether these appear as vasomotor changes or as involuntary movements.

This physiological law is in complete accord with the belief of many historians of language, independently reached, that all words ultimately spring from concepts or ideas based upon motor (muscular) activities (compare remarks on I, above).

The concept, thought, or idea and its verbal expression—the word—sum up the whole potential motor attitude of the individual to the actual corresponding reality; the motor attitude appropriate to the 'cat' is something very different from that appropriate to the 'lion'; and he whom a 'lion' does not stimulate into the motor attitude appropriate to the object would sharply pay the penalty of his lack of sensuous experience with the object denoted by that particular name; and one's success in dealing with the situation would doubtless be proportional not to the amount one had read or heard about lions, but to the amount of sense-experience one had gained with actual lions. The same is true with respect to geometrical concepts.

These truths have been emphasized with a view to mathematical education; for other branches the third statement would require modification, as therein (e.g. in moral, aesthetic and other species of mental activity) we have to deal with internal feelings as well as external sense-impressions; in mathematics the former may—for the present purpose—be safely neglected.

An important deduction from II, above, is: II (*a*):—Most thinking is done without language. This applies to the painter, sculptor, musician, experimenter, all artists and craftsmen—and above all to the child, and the more so the younger he is. The external expression of the thought (its own contents based as above stated on sensation and feeling) may or may not be verbal: if verbal, then a distinct scientific or literary discovery, communicable to others with like experience, has been made. The discovery, of course, may be great or small. Now, it is one of the functions of the teacher to stimulate the pupil to a development into verbal or literary expression of his sub-conscious and conscious experience, and language itself is the teacher's most effective

instrument thereto. But this very fact implies that the experience must be there first, or it cannot be developed into self-expression either verbally, or non-verbally (as in designing, geometrical drawing, &c.). The fit place of the verbal expression is as the summit and crown of the experimental process. Science and literature in individual and race follow sensuous experience as its verbal expression; to place the language (definition, &c.) first is to reverse the natural order by artificial authority of the teacher, and results in mere rote-work, lack of interest, and suppression of initiation and originality—is, indeed, to give the symbol without the experience symbolized.

The teacher cannot realize too clearly the profound truth that the substance, contents, matter (call it what one will) of ideas can never be created in the pupil's mind by mere spoken words, however carefully chosen. The spoken word is in itself nothing but so much sound, and signifies to the pupil nothing but so much sound unless the substance, matter, contents of the idea, symbolized by that word, already exists in the pupil's mind as a sub-conscious product created by previous physiological stimulation which resulted in sense-impressions. If this experience, lying latent or sub-conscious, is there, then the function of the spoken word is to raise this sub-conscious product to the level of consciousness. Thus the pupil becomes conscious of his own stores of experience, is assisted to create new mental connexions, to strengthen old ones, and in short to systematize his experience into a conscious tool available for his own further development. The substance of ideas, as such, is therefore not communicable from one mind to another. In speaking to others the condition that our words be intelligible is a reasonable degree of similarity of sense-experience in the sphere under discussion, in the pupil and in oneself. The greater the similarity, the greater our intelligibility: the less the similarity, the less our intelligibility, until in extreme (and too frequent) cases the dissimilarity of experience is so great that there is complete misunderstanding between pupil and teacher, and rote-work is the inevitable result.

A warning may be added here. The sight of a few pairs of intersecting straight lines, after a certain degree of sensuous familiarity with lines, may appropriately lead to the verbal truth—a summing up of one fundamental pro-

perty of the straight line—that two straight lines intersect in one point and one only. Here, in the summing up of a piece of sensuous experience, language *suc*ceeds to sense-experience. But the newly-gained language in its turn serves to sharpen the senses to discover new aspects of concrete things, otherwise blindly overlooked, and therefore such language now *pre*cedes the discovery of new sensuous experience: but this again simultaneously classifies and deepens the significance of the previous language, and demands new language for its own expression and summing up. The full significance of the above apparently simple geometrical truth not even a Newton could master, since it is literally endless. Thus the natural development of mathematical experience is a constant alternation of the abstract thought and the concrete sensuous experience, leading together to a rationalized experience based upon sense.

The order is, then, not a mere simple passage from sense to thought, from concrete to abstract (resulting, as this would do, in arrest of development), but a spiral of progress in which each new group of sense-impressions leads to the creation of a new concept (and new language in general, for language, as stated above, is not co-extensive with but less than thought, concept or idea), and in which each new concept thus gained is (consciously or unconsciously) applied to the development of new sense-impression. The two processes lead to fuller and fuller interpretation of the world around, and deepening significance of language itself. Faraday emphasized one aspect of this educational principle in the saying: 'The eyes see what the mind looks for'; parallel truths are applicable to all the senses: not only sight but touch, hearing, &c. And the complementary truth to this is the Pestalozzian principle that only what has been sense-experienced can be understood. Kant may be appropriately quoted to sum up the two aspects, neither of which any teacher can neglect with impunity: 'The senses without thought are blind: thought without the senses is empty.'

Long and persevering reflection upon this last maxim, embodying as it does perhaps the greatest discovery in psychology, combined with the serious determination to test its truth, and thus in a sense rediscover it for himself by observation of himself and his pupils, will profit the teacher vastly.

CHAPTER XI

ADDITIONAL ILLUSTRATIONS OF THE VALUE OF CERTAIN CENTRAL EDUCATIONAL PRINCIPLES IN THE TEACHING OF MATHEMATICS[1]

A typical development of a central geometrical truth.

THE teacher will observe throughout the lectures that the dominant stress has been laid upon geometry in the teaching of mathematics. The reasons for this are manifold: the main reason is that geometry forms throughout Mathematical Education that central developing portion round which nearly all the other branches can be naturally grouped. Of the many vices characteristic of much Mathematical Education, not the least is the separation of the science into a number of almost water-tight compartments—Arithmetic, Algebra, Euclid, Practical Geometry, &c. Yet the science is a true unity and should be so developed from the beginning, and Geometry is the uniting fluid that runs through the whole. This is clear both from the history of the science as a whole and from experience of teaching; and it is evidenced in the fact that number, position, shape, and size are four fundamental characteristics of all material objects, and, of these four, three are purely geometrical. It is, of course, necessary to have at times some substantial degree of isolation, as e.g. in the development of number as discrete; in the passage from particular to generalized arithmetic, and from generalized arithmetic to its symbolic form known as algebra—in all of which proper stress is laid on the actual operations, or the symbolical aspect of each, e.g. in such truths as (1) $2 \times 3 = 6 = 3 \times 2$; (2) $\frac{1}{2}$ of $\frac{1}{3} = \frac{1}{6}$; (3) $(a+b)^2 = a^2+b^2+2ab$, and so on; but even here the

[1] Notes of Lectures to Saturday Morning Classes of Teachers, Sunderland, 1903–4.

isolation should rarely be great or long continued. Thus, truth (1) should be supported by the geometrical intuition that the groups

$$\begin{array}{ccc} \cdot\ \cdot & \quad\text{and}\quad & \cdot\ \cdot\ \cdot \\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\ \cdot\ \cdot & & \end{array}$$

are equivalent, being merely differing positions of one and the same rectangular array ; truth (2) should be supported and illustrated by geometrical applications (lengths, surfaces and volumes), thus forming a basis for the more generalized forms of mensuration which naturally develop with such formulae as (3).

Thus the development will appear (in essence) as :—

Number.		Size and Shape : Position.
Arithmetic (Particular Number).		Descriptive Geometry (Shape and Position).
(As in (1) and (2) above.) ↓	Elements of Mensuration and Geometry of Position.	↓
Algebra (General Number).		Do.
(As in (3) above.)	Mensuration, Geometry of Position, Trigonometry, Analytical Geometry, &c.	

The aim is : (1) To lead up to a natural development of truths (the test of the 'natural development', as the phrase is here used, is either the essential stages passed through by our ancestors in the discovery and proof of the proposition or truth, or (equivalently) the order of development appropriate to the brain maturity of the pupil who is ready to assimilate the proposition or truth) : (2) To throw the subject of study into bold relief, wherein the truths severally gain such outstanding prominence, and are based upon such degrees and variety of evidence or proof as are appropriate to their importance in the science.

To effect this :—Let the teacher pick out the central truths and group the rest in a reasonably systematic

manner round these : develop a variety of proofs of these central truths and 'illustrate' them copiously both before (in leading up to them) and after (by way of producing conviction of their truth) their discovery, with quantitative measurements. The pupil should be able to get quickly down to the intuitional basis of any central truth, so that his knowledge of essentials may be refreshed continually at its source and his faith and clearness strengthened. Subordinate truths may safely be left to hinge on to the central truths. It is an indispensable art the teacher should cultivate assiduously, and one never perfected because always further developable—the art of throwing his subject into true relief. Mathematical science is, like other knowledge, a land of mountains, hills, plains and valleys—not a monotonous Sahara of sand waste.

In the following sketch the teacher is reminded that :—
(1) The initial practical problem of map-drawing led to the development of fundamental ideas on the determination of position : these developed ideas of shape and size : these, again, led to the systematic discussion of the fixing in shape and size of a triangle, thus leading to the discovery that three sides, two sides and the included angle, &c., suffice to determine a triangle.

(2) The question then naturally arose : Of the six elements (three sides and three angles) of a triangle, are any three sufficient to fix its shape and size ? The examination and classification of the six elements then follow ; leading to the substance[1] of Euclid I. 4, 8, 26, first part (the second part, viz. one side and two non-adjacent angles, e.g. A, B, a, —being touched upon and reserved for fuller discussion until the proposition now to be established regarding the angles is dealt with and wherefrom it can easily be deduced : this is not essential but advisable).

(3) Finally, the problem arises : Are the three angles sufficient to determine a triangle ?

The interest awakened in the study of geometry by this mode of development is astonishing to teachers accustomed to the Euclidian or other formally logical march. The rational grip of the truth, too, ultimately attained and the power to apply it are strongly developed.

[1] Not, in general, the mode of proof, unless it were by superposition as in I. 4, I. 26 (first part).

The teacher's function here is to develop gradually in the pupil some understanding of, and ultimately some power over, some of the methods by which such problems are attacked by a scientific investigator.

One simple and natural method for dealing with this problem is to draw a triangle, and measure the three angles: say $58\frac{1}{2}°$, $22\frac{1}{2}°$, $100°$. Some small error in actual measurement is inevitable, in general. Perhaps it is too much at present to expect the introduction of the simpler ideas underlying the Probability treatment of errors of measurement into School Mathematics: but the hint may be here inserted in accordance with the recommendation of Laplace, the great French mathematician of the eighteenth century. In

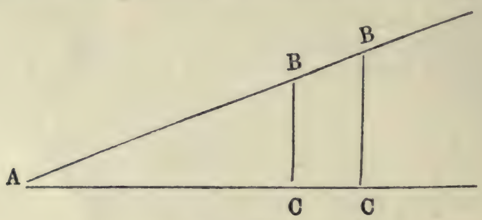


FIG. 30.

the present problem, when once it has been found that in an ideal triangle the three angles amount to exactly $180°$, the problem is to divide up this sum appropriately between the three actually measured angles, with consequent rectifications. The writer has himself often introduced simple ideas of Probability into elementary teaching and found them valuable. Doubtless some day the measurements of pupils will be sometimes combined in accordance with the 'weights' awarded to their respective degrees of skill. '*Probability is the guide of life.*'

Now, remove this triangle. Can you reconstruct it from the above data or measurements? Try. Start with $A = 22\frac{1}{2}°$. It is seen that there is no datum to fix the position of B. Clearly some length must be given.

This fact, that one length at least is necessary in the construction of figures (in size and shape) will, of course, be seen by some pupils previously, though not, probably, in this general form.

Though we cannot determine the size, can we determine some other property of the figure with these three angles ?

On trying to build up the figure, the pupil finds the third angle (approximately) formed for him by the mere construction of the other two ! What does this mean ?

This experiment should be repeated. Better still if each pupil takes a different kind of triangle to experiment with. Instead of a triangle constructed on the paper with pencil lines for boundary, the teacher will often find it convenient and valuable to have a stock of varying sized and shaped paper triangles, cardboard triangles, wooden triangles, and triangles formed of jointed rods. Similarly with quadrilaterals, or other polygons. These triangles will be found useful throughout the mathematical course and are strongly recommended.

Is a similar property true of the sides ? Try. No ! The three angles are then somehow connected, for any two fix the third.

Interdependence or Functionality.

Here the teacher should develop the idea, very fundamental, of interdependence—of dependent and independent variables. The idea of a variable is one to which, in mathematics, scarcely too much attention can be paid : the idea, moreover, should be inculcated early : its full development is, of course, absolutely endless ; the more reason that it be begun early before the mind becomes stereotyped with the idea of fixity. See that both the idea and the quantitative symbolization (viz. $A^\circ + B^\circ + C^\circ = 180^\circ$) are better developed. Thus, if $A^\circ + B^\circ + C^\circ = 180^\circ$, or, better, passing over into the usual symbols (as determinateness here gives place to variability and therefore justly demands a new type of symbol), $x^\circ + y^\circ + z^\circ = 180^\circ$, we have three distinct truths :—

$$\begin{cases} x^\circ = 180^\circ - y^\circ - z^\circ. \\ y^\circ = 180^\circ - z^\circ - x^\circ. \\ z^\circ = 180^\circ - x^\circ - y^\circ. \end{cases}$$

Copious numerical illustrations should accompany this ; but, of course, after the theorem has been discovered. Note the use of the concrete sign for degrees, x° (especially necessary in early attempts at generalized arithmetic, as above), and the gradual need for development of the logical truths

that one unknown requires one equation for its determination, two unknowns two equations, and so on.

The aim is now to find some functional relation between the three angles of a triangle.

Note that the equation must clearly be such as to give any angle uniquely in terms of the other two, as, geometrically considered, the construction of A and B, two angles, uniquely determines the position of C and $\therefore$ size of the angle at C. Ambiguous or multiple determination may be illustrated with equations like $x^2 - 5x + 6 = 0$: and geometrically with the well-known 'Ambiguous' case in the determination of a triangle by two sides and a non-contained angle.

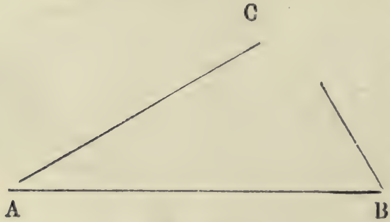


FIG. 31.

How can we find this numerical relation between the three angles of a triangle? One simple mode is actually to draw a number of varied triangles and tabulate the measured angles. Thus :—

$A°$	$B°$	$C°$
18°	71°	$89\frac{1}{2}$°
45°	$37\frac{1}{2}$°	98°

Does any pupil see any simple relation between A, B, and C?

If not, do not give away the secret, but suggest to your pupils some other mode of attacking this interesting problem.

Thus, if quantitative tables of this sort fail, inculcate in your pupils the methods :—

(1) Of trying extreme cases.

(2) Of considering special cases where the figure has some kind of obvious regularity.

(1) Suppose one angle becomes very small: what happens? Suppose one angle becomes very large: what happens?

What is the smallest angle, and what is the largest angle one can have in a triangle?

These questions lead inevitably, naturally, and simply to some properties of parallel lines, and the valuable notion is generated that parallel lines are lines meeting at infinity, or, more simply, that lines approach parallelism the more, the more distant becomes the meeting-point, or, that in the limiting case when one angle becomes zero the triangle transforms into a pair of parallel lines intersected by a third line. The idea of gradual transformation of one figure into another, the fluidity of figures, is of equal importance with the idea of an algebraical variable, of which indeed it is one aspect.

After a little trial it becomes clear that the smallest angle possible is as near to zero, and that the largest is as near to two right angles, or 180°, as one wishes; the remaining two are then as near to zero as is possible, in which case the triangle tends to become merely a pair of coincident straight lines. [Here it is of interest to point out that a pair of coincident lines may, by small changes continuously applied, be finally turned, through a triangle, into a pair of parallel lines intersected by a third line.] The hint will now be sufficient to lead to the discovery that $A+B+C$ is approximately 180°—an experimental result which should be tested by fresh cases.

The final result so far reached is :—So far as we can measure, the three angles of all the triangles we have measured amount together to approximately 180°, or two right angles.

Note the precise scope of this statement: the evidence so far does not warrant anything more. Experimental evidence of this kind does not and can never establish absolute definiteness and perfect equality. (See Chapter IV.)

(2) Should the teacher prefer to omit consideration of extreme cases, as above, or should the desired discovery not

yet be made, then the procedure might be this. Examine some regular kinds of triangles. Try equilateral triangles : right-angled triangles with two sides equal and so on. Here the sum of 180° is almost inevitably discovered : moreover the examination of these cases is an excellent preliminary to the 'intuition' treatment now to be described.

The questions now arise :—

Is this relation true for all imaginable triangles ?

Is it exactly true for perfectly drawn triangles ?

Intuitional Evidence.

At points like this arises obviously the need for some method richer than mere concrete measurement, and this principle the pupils should gradually grasp and master. So far, the procedure has been predominantly experimental—not wholly : for the experimental, intuitional, and scientific pass insensibly and successively into each other like the successive ages of man. Nor, rigorously speaking, has any geometrical proof ever been supplied in which any one of the three elements is wholly absent, any more than any mental state can be found in which will, thought, and feeling are not commingled. But any one element may be so predominant as to obscure the other two and justify us in naming the proof empirical, intuitional, or scientific respectively, for educational and classificatory purposes. We must now have recourse to predominantly intuitional evidence. The teacher may be briefly reminded that by intuitional is meant direct appeal to 'obvious' first principles without the intermediacy of long chains of systematic reasoning : and 'obvious' not necessarily to every one, but to the pupil at his present stage of development, and easily brought into consciousness after a sufficiently large experience in the experimental stage. Indeed, the experimental stage, supplying the sense-basis of the intuitional, may be termed the sub-conscious experience which the teacher can stimulate into conscious intuitions, so that the intuition is in one sense 'direct' (like a flash of insight), but in a deeper sense implies a large previous sense-experience for its birth

Here, too, we may wisely continue following the historical development of the discovery of and evidence for this central

geometrical truth. In the succeeding description this is, in essence, actually done.

Take any two identical right-angled triangles (such as I have here). Put them together thus, so as to make a— What? A rectangle.

Clearly any rectangle (by appeal to notions of symmetry: here occurs 'intuition') may split up into two identical right-angled triangles. Moreover, any two identical right-angled triangles can be constructed into a rectangle.

What does this suggest? Some brief questioning will bring out the facts:—

(1) The angles of any rectangle amount to four right angles. Note further appeal to 'intuition'. The axioms or postulates implied here (it is quite unnecessary to make explicit mention of them at this stage) are that rectangles exist, and that each angle is a right angle, &c. This 'intuition' can, of course, only be successfully appealed to if the pupils have had sufficient sense-experience of rectangular objects in ordinary life, e.g. book-covers, sheets of paper, surfaces of tables, walls and ceiling of rooms, &c., &c. But it is the teacher's function thus to utilize to the utmost all this sub-conscious sense-experience, and bring it to conscious application in geometry for the development of deeper truths such as the above.

(2) Every right-angled triangle is half a rectangle.

What follows from (1) and (2)? [Of course, with figures.] Clearly, that the angles of any right-angled triangle together amount to the half of four right angles, i.e. to two right angles.

Impress upon the pupils (1) the universality of this conclusion, (2) its perfect quantitative accuracy: both characteristics of this new mode of proof sharply to be distinguished from the particularity and approximate nature of the experimental method.

Strictly stated, it has now been shown:—

I. If a perfect right-angled triangle could be drawn, it would always have its three angles equal exactly to two right angles.

Corollary:—The two acute angles of any right-angled triangle always amount to one right angle. After the hypothetical nature of all these propositions has been reasonably well mastered by the pupil, it becomes un-

necessary and cumbrous to repeat them in the form 'If a perfect,' &c. Nevertheless, this essential difference between absolute and perfect though hypothetical proof and the experimental proof must finally be made clear and firmly impressed upon the pupil's mind.

Having arrived at this point, copious quantitative exercises and simple logical riders should be given.

Transcendental Geometry.

In the elementary treatment of this subject modern transcendental geometry (begun by Gauss, and continued by Bolyai, Lobateschwky, Beltrami, Riemann, Helmholtz, Clifford, and others) has no direct bearing. The historically and logically simplest view has been adopted upon which to construct the first branch of Comparative Geometry—the postulate (or its equivalent) that through a given point one straight line (Euclidean) and one only can be drawn parallel to a given straight line. But for the professional teacher of mathematics and of logic the researches of modern transcendental geometry are of the deepest significance, and should directly influence their teaching of the highest forms of a secondary school where the pupils show capacity for philosophical thought.

The justification for the adoption of the conventional attitude in early mathematical teaching lies in the educational principle that the nature and rigour of the evidence must be appropriate to the capacity of the pupil ; now transcendental geometry is certainly not appropriate to the capacity of the beginner ! Moreover, our historical principles also warrant this procedure. No system of truth—and, therefore, of geometry—is ever perfected : that is the most perfect for us which is best adapted to our intelligence.

Further Experiments.

Before leaving the predominantly experimental stage, the two following experiments may be recommended. Each is capable of such further development as may be deemed advisable by the teacher.

(1) Test the property just discovered by cutting off the corners of a paper triangle, and placing them together, so as to fill up half the space round a point, thus amounting to 180°.

(2) Take any paper triangle ABC. By folding, obtain a crease BD perpendicular to AC. Then fold A on to D, C on to D, and B on to D, so that the figure assumes a rectangular shape $PQRS$. Then it will be found that A, B, C, together, just appear to fill up two right angles.

'Just appear to' means 'so far as the eye can judge': the evidence is merely experimental: it is not a general proof, however many triangles one takes. Compare the above

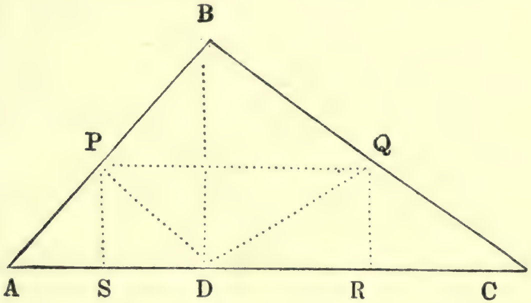


FIG. 32.

remarks on universal or general truth (see Chap. VIII, p. 102). Of course it is easy subsequently to expand this evidence, by reasoning about the general properties of perpendicular bisectors, into a satisfactory general proof: this, indeed, in its proper place, is an excellent exercise in reasoning. Note that simple ocular evidence is given in this experiment to suggest that the rectangle above obtained is the maximum inscribable in the original triangle. For try others, and all will be found less than half the triangle.

Continuity of the Three Types of Evidence.

The next step is to extend this truth I (page 131) about right-angled triangles to all triangles, if possible. Here we alight upon another of those fundamental configuration properties of a triangle which are only passed over in silence in elementary geometrical teaching because they seem so very obvious to the teacher; but the truth here referred to is so important as to justify a special statement of it. The ques-

tion is: What relation has any and every triangle to a right-angled triangle? Can every triangle be divided up into two right-angled triangles? Yes, by folding. Thus, even if the triangle be obtuse-angled, we can still fold it so as to produce two right-angled triangles.

A remark is advisable here. Doubtless, to prove this fact in the Euclidian or scientific manner, we should find ourselves first compelled to prove the very proposition we are desirous of establishing; for a few trials soon disclose the fact that two angles at least of a triangle must be acute-angled in order that the folding within the triangle may be practicable. But, even then we should—strictly speaking—need to rely somewhere or other upon some unproved intuition which we call a postulate or axiom: I say 'strictly speaking' because doubtless we should quietly assume this postulate without being conscious of the fact. For elementary purposes, therefore, as the above fact respecting the folding is a truth which intuition very readily grants, when based upon some reasonable amount of experience in folding triangles under the given conditions, we may as well, in accordance with our principles, utilize it here. It is precisely in the skilful use of such simple intuitions that the teacher will show his fitness for the work. The pupils will, of course, before assenting to the possibility of every triangle being decomposable into two right-angled triangles, test the truth by a variety of cases. It may be objected here that however many cases be tested, the pupil is not justified here (any more than in measuring the angles previously) in drawing, from merely particular cases, a universal proposition. Two replies may be made to this apparently sound objection.

First, that a genuine intuition differs from a mere experimental measurement precisely in the fact that, though experience of particular instances is a necessary preliminary in both processes, a limited number of instances (varying in different cases according to the intelligence of the observer) suffices to establish an intuition, not as being sufficient evidence in themselves, but as unconsciously suggesting to the creative imagination the bringing into operation of its own peculiar mode of evidence [resulting in a flash of insight—an intuition]; while no number of instances, however large, can establish universality in a truly quantitative experiment

as such. In the former case (intuition) the truth obtained is merely suggested by the instances and goes legitimately far beyond them: in the latter (quantitative experiment) the truth is entirely based upon them, and simply sums them up justly, giving neither more nor less than they themselves singly contain. One operation is largely chemical; the other is largely mechanical. One requires originality and insight: the other a little common intelligence. Of course, instances doubtless could be selected, and will occasionally occur in teaching, wherein the thoughtful observer will find it difficult to separate the element corresponding to mere experience from the element corresponding to creative imagination: one may even go further and maintain that in no single case can the two elements, in an ultimate analysis, be found absolutely separated; but this fact does not prevent us for practical purposes from drawing the distinction, which is justified in the vast majority of cases by the actual predominancy, at least, of one or the other element. It is the existence of such subtleties in human reasoning—which can ever in obedience to the desire for unification invent continuity in its reasoning about nature even where continuity is not actually to be found by the senses—that warns us as teachers to regard any so-called division of human evidence into distinct categories of experimental, intuitional, and scientific as working arrangements merely, infinitely convenient, but easily passing over the one into the other, and therefore, not to be made a fetish of. The teacher will probably find he uses this instrument—like any other—most wisely and efficiently when his familiarity with its use and his understanding of its basis are so great that he at length employs it unconsciously—automatically selecting the right employment of it in particular instances: but let him beware that the automatism does not precede instead of following reflection upon the meaning of the method.

A second reply may be given to the above objection. When the teacher has sound reason for thinking that the pupils are not able to acknowledge truthfully the universality of the intuition (as possibly in the present case—only experience can decide) and do not feel its universality, then the evidence may lead to the modified form of statement:—So far as we can imagine at present (or more limited still—

so far as we can see at present) every triangle can be split up into two right-angled triangles. This is a sufficient basis to work upon. But possibly, in so very exceptional a case, it would not matter should the *ipse dixit* of the teacher intervene and assure them that further experience will convince them of its universal truth. Should this procedure be objected to, it may be replied that in that case all intuitional (and scientific) statements should be put into the more guarded form, as we can never be quite sure that fresh insight and experience may not produce exceptions to our statements. Clearly, common sense is needed here, and he who has not a reasonable fund of common sense should not be teaching geometry—or anything else.

It may also be noted that an assumption closely similar to this is made by Euclid himself.

Upon this fundamental fact—(II) that every triangle can be dissected into two right-angled triangles—rests almost the whole of elementary trigonometry and the greater part of mensuration.

The folding of acute-angled triangles to produce this dissection may be used to suggest the beautiful theorem —of which a proof can be afterwards supplied—that the three perpendiculars from the vertices on to the opposite sides of a triangle meet in one point.

Returning to the problem in hand, we have now to lead our pupils on to the discovery of the general proof of the angle-sum property of triangles. I use the phrase 'angle-sum property' as a brief description of the property that the three angles of a triangle together amount to two right angles. It is convenient to have a short descriptive name for all fundamental truths in geometry. Where the name of the original discoverer is known this might appropriately be used : in the present case there is some reason to think that Thales, the Greek mathematician and philosopher, was one of the earliest to supply a tolerably satisfactory logical proof of the generality of the truth. Doubtless the Egyptians knew of its existence as an experimental fact ages before the time of Thales.

It will not be found difficult to deduce the general property from the truth marked I (page 131), and the truth marked II (page 136).

The reasoning, briefly stated, is as follows :—

Divide any triangle ABC into two right-angled triangles (by drawing from A a perpendicular, AD, to BC, A being a suitably chosen vertex : this may be done either with instruments or by simple folding, or, if preferred, 'hypothetically performed ').

Then $\angle BAD + \angle ABD$ = one right angle (Truth I Cor. above)
$\angle DAC + \angle DCA$ = one right angle do.
$\therefore \angle BAD + \angle ABD + \angle DAC + \angle DCA$ = two right angles.
Now $\angle BAD + \angle DAC = \angle BAC$
$\therefore \angle A + \angle B + \angle C$ = two right angles.

Clearly, similarly reasoning can be applied to every triangle, and the property is seen to be universal and absolutely accurate, for perfectly-drawn triangles.

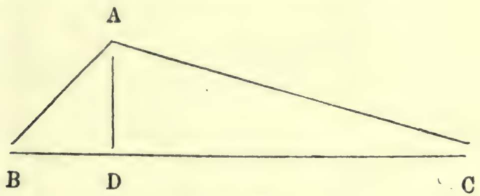


FIG. 33.

The teacher will note, in this mode of proof followed to establish the angle-sum property, both repeated intuitional appeal and the beginning of systematic logical form, i.e. the beginnings of scientific evidence, in which the truths presented are shown to be interdependent and to form chains of premisses and conclusions.

Thus the total line of evidence is :—(see 1, 2, 3, ... 8 below).

1. There exist figures known as 'rectangles' (which possess a certain kind of symmetry).

The evidence for this is partly experimental, the contribution of sense-perception, and partly intuitional, the contribution of imagination. The pupil has been accustomed to rectangular shapes all around him, and can actually construct rectangles in several ways.

It is strongly recommended that, amongst the many methods which might and should be used, one be by paper

folding. Thus, if X be any irregular piece of paper (the printer for convenience has taken a circle: Fig. 34 is only given for illustration), by making any crease, say AB, and then folding at any point this crease along itself, a second crease COD is made, clearly perpendicular to the first (as the two adjacent angles fit, they are equal, and each is clearly half a straight angle, and therefore each a right angle.)

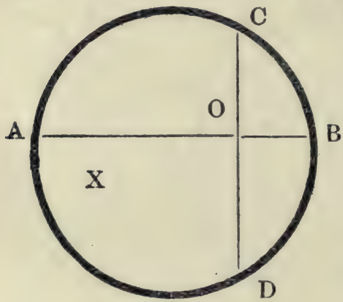


Fig. 34.

We have now got one corner, say AOD, of our rectangle. A second corner, OPQ (Fig. 35), is easily obtained similarly; then a third, PRS, similarly, and it will now be found by the pupil that the fourth and last corner is thereby determined for him, viz. RSO, and that S is also a right angle.

That S is really a right angle may be evidenced by superposition combined with intuitional ideas of symmetry when the eye and mind become familiar with the whole figure. I venture to think that if a reasonable amount of actual handling and seeing and construction of rectangles has been done by the pupil, very little or no additional evidence is required to convince the pupil that figures known as rectangles exist having the properties desired.

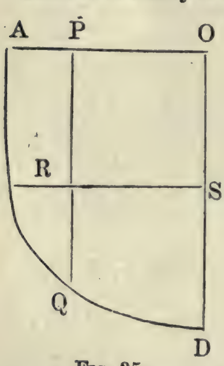


Fig. 35.

2. Every rectangle is decomposable into two identical right-angled triangles. [Evidence partly experimental and partly symmetry-intuitions.]

3. Any two identical right-angled triangles can be made into a rectangle. [Evidence as before.]

4. The angles of a right-angled triangle amount to exactly half of those of a rectangle.

5. Therefore the angles of a right-angled triangle amount to two right angles.

6. Therefore in a triangle with one right angle the remaining angles are equal to one right angle.

7. Now every triangle is decomposable into two right-angled triangles. [Intuition and experiment.]

8. Then, as before, see page 137.

The teacher here will observe that the whole procedure is originally predominantly experimental, appealing to sense-perception mainly, that it afterwards becomes predominantly intuitional, and that, taken as a whole, it has the note of a scientific system. But, throughout, none of these elements absolutely vanish. In some degree, they are all three present in every conclusion, even the simplest.

Experiment, Intuition, and Science.

Even at the risk of wearisome iteration, the writer ventures to emphasize again the importance of the experimental element as: (1) forming the necessary preliminary to further development; (2) giving the material to sense-perception in showing the actual constitution of space; (3) establishing particular and approximate truth; (4) suggesting general and absolutely accurate truth; (5) acting at any stage as a court of appeal for testing and verification of reasoning, for refreshment of faith, for development of new space-material for discussion, for application of final reasoned truths.

While the rôle of the intuitional is: (1) to exercise the imagination in the conversion of experimental suggestions into accurate and general statements or truths; (2) to interpret the isolated deliverances of the senses into meaning and significance by drawing out into consciousness and linguistic expression the principles inherent in our use of the senses to understand the world; (3) to give vividness and the conviction of truth to one's interpretation of the sense-products, i.e. to affect the feelings concerned in the development of experience into knowledge; (4) to act throughout as a rational basis of support to which rapid and direct appeal may be made to refresh the sources of our rational conclusions; and (5) in fine, to act as the main medium in converting mere sense-experience into knowledge and skill in applying it to the world around.

Finally, the rôle of the scientific (in so far as it is superior to and distinct from the last) is to show the formally logical interdependence of our conclusions, to exhibit our knowledge

as a more and more systematized and consciously ordered whole. Its ideal, never reached, is the construction of a system of truths in which, from a limited set of postulates or axioms (by the help of the ultimate assumptions common to all thought) peculiar to and characteristic of the particular science in question, all the remaining conclusions in the science can be deduced by rigorous chains of reasoning. The grandness of this ideal is obvious. But a careful consideration of it shows us, as teachers, that we require much common sense to know to what degree our reasoning should be scientific. We have ever to remember that the more scientific our form of reasoning—that is, the longer the chain and the less the direct appeal to sense and intuition—the more difficult is the reasoning to follow. Here, as elsewhere, the teacher must use common sense, and remember that the nature and degree of rigour of the proof or evidence must, if it is to be assimilable, be appropriate to the maturity of his pupils' intelligence and experience.

But if this principle is sound, we violate it equally whether we develop scientific procedure too early or postpone its development too late.

Epitome of the Evidence.

Once the various steps in the evidence have been grasped, it is highly important to bring the whole evidence, if possible,

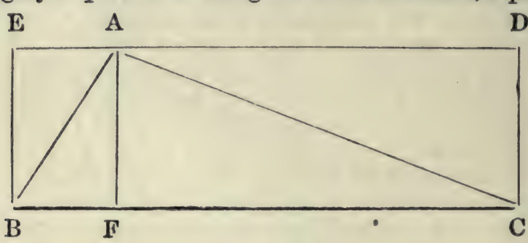


FIG. 36.

into one view and in one figure before the pupil, in order to produce a feeling of conviction of the truth of the general conclusion finally reached.

Thus :—Starting with any triangle ABC, build it up into the rectangular figure $EDCB$ with AF perpendicular to BC. Then (slightly changing the elements of the proof, to obtain

a rapid view of the whole) ABF is identically equal to BEA (each, half the rectangle EF), therefore $\angle AFB + \angle FAB =$ one right angle.

Similarly $\angle FAC + \angle FCA =$ one right angle.

Therefore the three angles B, C, A, equal together two right angles.

[If desired, it may be observed that angle $BAF = ABE$ individually, &c.]

Historical Aspects.

The whole course of evidence now described coincides in essence with the course actually traversed by our ancestors in the discovery of this general truth. Predominantly experimental among the early Egyptians, predominantly intuitional among the earlier Greek mathematicians who studied the Egyptian results, and, finally, predominantly scientific (and with a very high degree of scientific evidence) amongst the later Greek mathematicians, this truth is in quite recent times undergoing a complete re-testing and re-investigation. An attempt is being made to isolate the purely spatial empirical or experimental elements embodied in it from those elements contributed by the reason working upon all experience, and to effect this solution appeal is necessarily being re-made to sense-experience. Thus we have the instructive and remarkable phenomenon of a system supposed to have been discussed with rigorous and final logic, and to have been placed beyond question for over two thousand years as a closed product of science now proving full of hitherto unrecognized assumptions and falling into place as only one, if the most important, species of Comparative Geometry.

Science never reaches its goal, and even geometrical truth, once the stronghold of the idealist, has now revealed evidence of empirical limitations once inconceivable.[1] We

[1] The existence of elements, ultimately ideal, is not thereby disproved. Their constant historical recession, however, does appear to indicate the extreme probability that their nature is such as does not admit of isolation and precise verbal formulation valid for all times. The decisive factor may be the character of man's universe of experience. Expanding, whether in race or individual, it is the source of perennial conflict between idealism and empiricism: stagnant, it is the bourne of their union. Conflict of ideas is the price paid for intellectual growth. (See also Chapter XXII, on 'Axioms'.)

cannot dispense for long with the concrete or with intuition, even in matured science—much less in elementary education. This bit of history is surely significant for the teacher.

It may be added to this historical note that there is some ground for holding that, of the many methods by which the early Greek mathematicians probably approached the evidence for the universality of this truth—the angle-sum theorem—not the least notable was the observation of rectangular tiles, in which each half tile is a right-angled triangle identical with its neighbour.

Finally, it is interesting to know that Clairaut, a great French mathematician of the eighteenth century, observed that the length of the reasoning by which Euclid arrived at this truth in I. 32 is highly objectionable in elementary teaching, and postpones too long the use of this very fundamental property of a triangle. So he proposed in place of Euclid's chain (from I. 1 to I. 32) a proof whose elements are identical with that given above, and thus traversed, without being conscious of the fact, the very steps which the early Greeks probably followed before Euclid, and thereby applied also the historical principle in education here emphasized so repeatedly. The proof above given is hence sometimes known as Clairaut's proof.

Applications of the Proposition.

After the general truth has been established, it is perhaps hardly necessary to emphasize the absolute necessity of practising the pupil in the application and understanding of it by various kinds of exercises. These should be of many varieties, and may be conveniently classified as follows (the number of each advisable will depend obviously on circumstances, and should be reasonably proportional to the central importance of the truth in question) :—

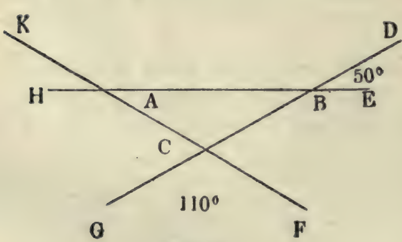


Fig. 37.

1. Simple quantitative applications, which may be verified by measurement, and therefore involve also generally construction with instruments; e.g. construct a figure of three intersecting straight lines (as in diagram) such that $\angle DBE = 50°$ and $\angle GCF = 110°$. Then calculate the remaining 10 distinct angles in the figure by use of the angle-sum theorem just established. Finally verify the calculations by actual measurements on the figure accurately constructed (i.e. verify as nearly as possible), or :—From a merely freehand qualitative figure (i.e. not drawn to scale) calculate the remaining angles and subsequently construct the figure to scale and verify the calculations.

N.B.—A few exercises of this sort will soon suggest that a triangle (and the above extended triangle) is a 2-data figure, as regards its shape, which (in this case) is equivalent to saying that when two independent angular data are given the shape of the figure is thereby determined. Why will the data ABC and ABD not suffice ? Name as many pairs of data as you can which will suffice to determine the figure in shape.

What else is required to determine it in shape and size ?

Construct a figure like the above, given that $AC = 5''$, the sum of the external angles at A and $B = 240°$, and the sum of the external angles at B and $C = 210°$.

N.B.—It will soon become obvious that when the shape of a figure is known, the size is determined directly we know the length of one linear element in it.

It is important to have sub-conscious experience (in the way of plenty of concrete measurement) of this truth—and similar general truths—before any attempt at formal proof is made.

2. Practical applications (e.g. to simple surveying, manual work, mechanical experiments, art, &c., &c., according to the curriculum and age).

E.g., to find the distance between two inaccessible objects : to draw a map of some group of objects, &c. All such problems may be made to hinge upon some application of the angle-sum property of a triangle if desired.

3. Applications requiring further generalization and stimulating the discovery of other truths (which may or may not be very important in themselves, but are at least interesting as self-discovered).

Under this head may be placed exercises which are themselves important truths: so that each newly-discovered truth may be made the basis and the preparation for subsequent general truths which the teacher proposes to bring forward.

As types of this species of gymnastic may be mentioned :—

(*a*) Find a quantitative relation between an external angle of a triangle and the interior angles ? (The exterior angle equals sum of two interior and opposite angles.)

[If this is not seen at once—a simple corollary to the original proposition—let the pupils tabulate a number of particular triangles with this end in view, and no difficulty should now be experienced in discovering the desired relation.]

(*b*) The old Euclidian type of exercise—known familiarly as a 'rider', and given in the form of some general property to be proved formally from the original proposition.

This form of exercise is, of course, most valuable, and is apt to be unduly neglected in some modern extreme forms of experimental teaching.

(*c*) The discovery of properties which can be generalized easily.

E.g. There is a well-known theorem connecting the angle between the bisectors of two adjacent angles of a quadrilateral and the sum of the remaining two angles of that quadrilateral. It will be found an interesting and valuable exercise to give many of the usual riders (such as this) not in the form of some stated theorem which is given, and of which the formal proof is required—modern methods will make it more and more difficult, by the way, to demand this form of problem in examinations when the order of development has been different for different examinees—but in the form of some property to be discovered. (See also (*a*) above.)

In the present case the exercise might be given thus :—

Construct any quadrilateral. Bisect two adjacent angles. Find some simple general angular relation between the angle at which the bisectors cut and the remaining angles of the quadrilateral. Generalize your result for a polygon of *n* sides, and finally test the truth of your general theorem by application to a triangle. [Of course, these questions need not be all given at once—that will depend upon circumstances.]

To develop his pupils' intelligence, initiative, and interest in dealing with problems such as this, the teacher is asked to consider carefully the following suggested procedure which the writer has found valuable in his own experience.

Accustom your pupils in general to start with actual numerical cases (with of course a preliminary freehand trial figure showing the nature or configuration of the figure to be constructed). Let them actually construct a variety of quadrilaterals $PQRS$, bisect angles at P and Q, and tabulate the results along with the remaining angles of the quadrilateral. If N° is any number of degrees (within certain obvious limits) one pleases, they will soon find that one of the remaining angles, say R, may always be denoted by N°, in which case when the angle V (between the bisectors) is, say 50°, S is 100° − N° : if V is 40°, S is 80° − N°, whence it soon becomes obviously suggested that if V is M° then S is 2M° − N°, and ∴ that $R+S = 2V$.

However, this result suggested by experimental measurements (the above relations are not perfectly accurate, only approximate, remember, when actual measurements are taken without prejudice from knowledge of the general result) has to be proved universal and accurate for our ideal quadrilaterals.

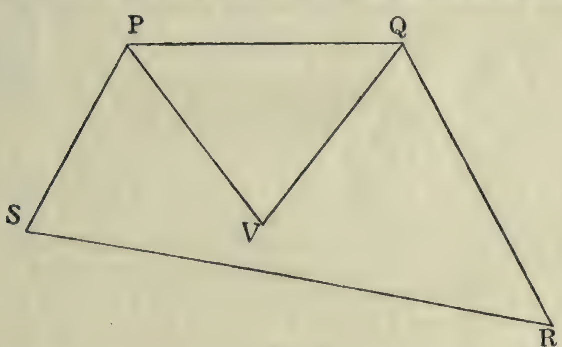


FIG. 38.

How is the pupil to proceed now ?

Accustom him to state in form of an equation quantitative properties of his figures. In the present case he knows and should be able to state that :—

$$P^\circ + Q^\circ + R^\circ + S^\circ = 360^\circ$$
and that $\frac{1}{2}P^\circ + \frac{1}{2}Q^\circ + V^\circ = 180^\circ$
or $P^\circ + Q^\circ + 2V = 360^\circ$

What follows therefrom ?

That $P+Q+R+S = P+Q+2V$

(dropping, when familiar, the symbol for degrees).

Therefore, $R+S = 2V$.

Hence: the angle between the bisectors equals half the sum of the remaining quadrilateral angles.

Or: the pupil may often proceed with advantage (as a mode of attack) as follows :—

There are, in all, nine variables (angles) in the complete figure, viz. :—SPV, QPV, PQV, RQV, S, R, and V, with SPQ and PQR.

But, connecting these nine variables we have six relations, viz. :—

$SPV = VPQ = \frac{1}{2}P$
$PQV = RQV = \frac{1}{2}Q$
$QPV + PQV + V = 180^\circ$ and $P+Q+R+S = 360^\circ$

Therefore, we can (in general) reduce these nine variables to three variables, so that choosing these three as independent and the other as, therefore, dependent magnitudes or variables, we shall have (selecting P, Q, R as the three), finally from the above relations :—

$$\left.\begin{array}{l} V = 180^\circ - \frac{1}{2}(P+Q) \\ S = 360^\circ - \quad (P+Q+R) \end{array}\right\} \begin{array}{l}\text{whence, eliminating } P+Q, \\ \text{we get easily } 2V = R+S. \end{array}$$

The geometrical aspect of this reasoning should be made clear: that there are three independent variables at our choice may be easily seen from the actual construction of the figure, which is a 3-data figure (in shape). One would, of course, take a much simpler example than the above to start with.

The next questions that should suggest themselves to the pupil (I take this as a type of example and method only) are:

Is this truth general for all polygons ? If not, what modification is required ?

Try the pentagon. Here we find (V = angle between bisectors of P and Q, and P, Q, R, S, T being the pentagonal vertices) :

$(P+Q)+R+S+T = 540^\circ$
$(P+Q)+2V = 360^\circ$
$\therefore R+S+T = 2V+180^\circ$

Generally, with N-sides, we get (A, B, C, D....N being

the vertices of the polygon and V the angle between bisectors of A and B),

$$A+B+C+D\ .\ .\ .+\mathrm{N} = (2n-4) \text{ right angles.}$$
$$A+B+2V = 4 \text{ right angles.}$$
$$\therefore C+D+\ .\ .\ .+\mathrm{N} = 2V+2\ (n-4) \text{ right angles.}$$

Put this into ordinary language.

Test it by application to a triangle, where N = 3, and also when N = 4. [For a triangle $C = 2V - 180°$.]

Parallels :—

The further treatment of parallels may be conveniently taken after this fundamental angle-sum property of a triangle.

I speak of the 'further' treatment, as figures illustrating and ideas connected with parallelism will inevitably have been developed to some extent in the Kindergarten years (in paper-folding, &c.), and also later. Here we have reached a point where these sense-experiences can be appropriately rationalized and systematized. It will be advisable first to retrace old ground by directing attention to the fact that parallel lines indicate sameness of direction, and to give a number of quantitative measurements (with set squares and paper-folding, with use of protractor for measurement of angles) upon the eight angles formed by the intersection of a pair of parallel lines by a third line—and, if necessary, also with the eighteen angles formed by the network consisting of three parallel lines, each cut by a pair of parallel lines. Such exercises will very rapidly familiarize the pupil with the kind of pair-equalities springing from such figures, and simultaneously with the names alternate, interior, and opposite, &c., angles. Not until these names have become thoroughly familiar by actual quantitative application should the formal proofs be undertaken of the universality of the various truths descriptive of the properties of systems of parallel lines intersected by other lines.

Of course, after a certain number of these measurements, the conviction that the alternate angles are always equal when the lines are really perfectly parallel and straight will inevitably arise under the stimulus of good teaching, for even a reasonably clear idea of direction applied to a pair of parallel lines suffices to produce this conviction after reasonable familiarity with angles and lines.

The formal proof of the properties of parallel lines may be now conveniently derived from the angle-sum property of a triangle.

[It is recommended that jointed models be used, as well as drawn figures, to illustrate the truths now in question. Thus, a pair of rods, freely jointed, each to a third, say at P and Q, so that QR and QP, say, being held fixed, PS may be revolved round P, from original coincidence with PQ through various points on QR further and further distant from Q until the position of parallelism to QR is reached, and then further round on the other side until PS performs a complete revolution round P. Such experiments give a clear conviction of continuity, and surpass almost any other kind of illustration in vividness and interest. Jointed rods may be strongly recommended throughout the teaching of mathematics, and every school and college should have a good supply of them. With the above system one may perform almost numberless experiments. Another useful

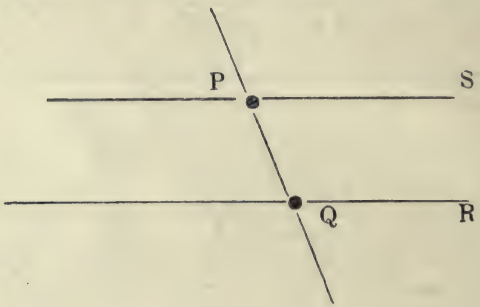


FIG. 39.

movement is, keeping QR and PS parallel, to separate them until the transversal PQ becomes perpendicular to both.]

Ask your pupils: What does the angle-sum theorem become when one angle becomes unlimitedly small ? (Illustrate by above jointed figure and by drawings.) It is easily seen that we derive the property that if a straight line intersect two parallel lines, then the two interior angles (on either side) together amount to two right angles.

[For if one angle becomes zero, the lines become parallel, the triangle becoming a figure consisting of a pair of parallel lines intersected by a third.]

What, in this extreme case, becomes of the corollary that the external angle of a triangle equals the sum of the two interior and opposite angles ? [Illustrate again.] ANSWER:

This truth becomes converted into the truth that, if two parallel lines are cut by a third, then the alternate angles are equal.

And so on for the other properties. The converse may be (as in Euclid) easily proved by a 'reductio ad absurdum'. Note that parallel lines must be in the same plane: contrast with lines in different planes, which never meet, however far produced, and yet are not parallel. The tacit assumption that all lines discussed in elementary geometry are in the same plane (or co-planar) should no longer be made (as in Euclid I, II, III, IV, V, VI), but repeated excursions made into solid geometry, to keep the mind (geometrically) flexible. Of course, (as in Euclid) since the properties of parallel lines are ultimately bound up with the angle-sum property (I. 32), any of these (properly selected) may be proved first and the rest deduced therefrom (assuming, in any case, certain postulates), but the course recommended above appears, for many reasons, advisable, though here as elsewhere, *quot homines, tot sententiae.* The above procedure is very clear, definite, and systematic, puts perhaps the most fundamental property characteristic of space first, and illustrates well the powerful principle of continuity as developed in modern geometry.

Size, shape, and position :—

The whole subject may be fruitfully treated also from the point of view of Geometry of Position.

[The fundamental fact in position here assumed is :—To fix the position of any rigid plane figure, confined to motion in a plane coinciding with its own, requires three and only three measurements, in general. The proof is easy. Fix one point of it (by driving a nail through it on to the plane). This requires two measurements (or co-ordinates). But the figure can still rotate round this point. If, therefore, in addition, we knew the angle some line in the figure makes with one of our co-ordinate axes, we could clearly fix the figure completely. Thus three data in all are sufficient, and are clearly necessary also, in general. This proposition, so fundamental in Geometry of Position, will again occur in its proper place in next Chapter.]

Thus, to fix a triangle in size, shape, and position, we require six measurements, viz. two for shape, one for size, and three (see above truth) for position. Also we may view the matter otherwise, thus :—

Suppose one vertex P is to be fixed :—two measurements needed. Two more for the second vertex Q, and still two more for the third vertex R. That is, six in all. But thereby we have clearly, in laying down the positions of the three vertices, fixed completely size, shape, and position simultaneously, for we have only to join P, Q, and R with straight lines, to get the triangle we desire.

Quantitative exercises, with specified data leading to actual instrumental constructions, should be given here.

Again we could fix each straight line in turn (two measurements each, viz. the two intercepts on the axes), i.e. six altogether.

Further : A pair of parallel lines cut by a third forms clearly a 2-data figure in shape and size, for the distance between the parallels fixes size, and any one angle fixes shape. To fix position requires three data more. Therefore five data are necessary to fix the figure in size, shape, and position.

As before, with a triangle, this may be elucidated otherwise by fixing each line separately, viz. two data for the transversal, two more for the first parallel, and one more for the other (we know already it is parallel to the first, and, therefore know its direction when the direction of the former is determined).

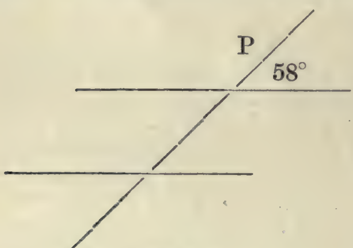


Fig. 40.

In all these positional problems, real quantitative measurements must be given, so that the actual figures may be constructed, sometimes with jointed rods, sometimes with pencil and paper.

For example : if P is 58°, calculate all the other angles, then construct the figure and verify your calculations.

Additional proofs of the Angle-sum Property of a triangle.

In the case of properties so fundamental as this it is advisable to supply (or suggest) a variety of proofs, each, so far as possible, independent of the rest, so that the mind may easily dig down into first principles when wishing to

recover its faith in the accuracy of its conclusions, and understand the source of its discoveries.

The following is a brief epitome of Hamilton's well-known proof :—

Lay a rod along one side QR of the triangle with its pointed end to r : revolve the rod round R from QRr into coincidence with RPp. Now revolve round P into coincidence with PQq, and finally round Q into coincidence with the original side QRr. In the first, second, and third movements of rotation, the rod revolves respectively through the three external angles. Seeing that the rod is in its original position, it must have revolved through a whole revolution, or four right angles. Therefore the three external angles of the triangle also amount to four right angles.

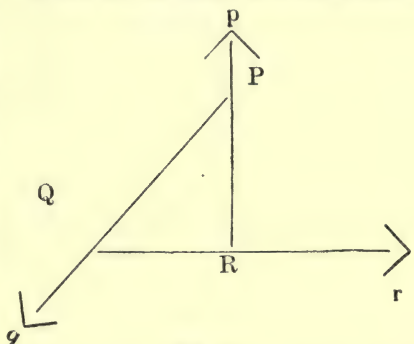


FIG. 41.

Whence it easily follows that the internal angles amount to two right angles.

It will be found necessary, somewhere, to translate or move the rod along in its own direction in order to obtain precisely the same position from which it started (unless the rod is a homogeneous infinite straight line, in which case subtleties spring up much too deep for elementary teaching). Hence, in this proof, the postulate has to be granted that translation and rotation are independent. It can be easily seen by a teacher acquainted with spherical geometry that this proof would lead to false results if applied to a spherical triangle, showing that translation and rotation may be independent in a plane surface, but cannot be on a spherical surface.

Envelopes, Curves, and Tangents.

Hamilton's elegant proof may clearly be extended to the external angles of any rectilinear or (finally) curvilinear polygon which is wholly convex to the interior. This

generalization should indeed be suggested, and the opportunity taken to instil the idea of a tangent to any curve as the limiting position of a secant.

Thus (briefly) if we imagine a polygon, whose sides are infinitely small and infinitely numerous, then, by emphasizing or throwing into relief the vertices, we conceive of a curve, in general, as the limit of an infinite number of infinitely near points. If, however, we emphasize the sides (each being extended both ways) of our polygon, then we may equally justly conceive of the curve in general as the envelope of an infinite number of infinitely near lines, each of which is a tangent to the curve. Each aspect should, from time to time, be presented in the elements of geometry, especially in connexion with the circle.

The intermediate view regards the curve as the limit of the conventional form of polygon itself where the sides are not extended and where, therefore, each becomes ultimately small without limit.

Finally, we may note the two elegant forms of proof depending upon the figures :—

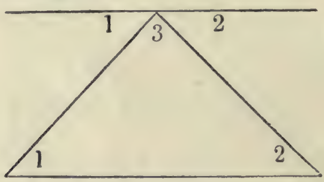


Fig. 42.

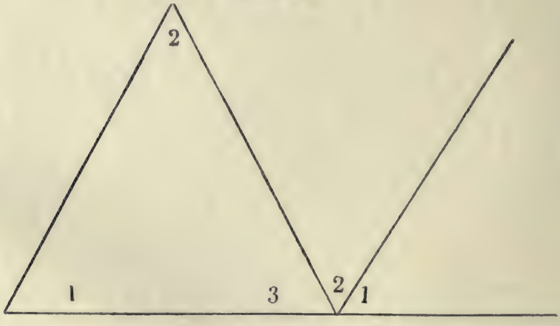


Fig. 43.

which (Euclidian) are too well known to need recapitulation, but both of which, of course, presuppose a previous treatment of the properties of parallels.

A modification of the Hamiltonian proof consists in revolving the rod through the respective internal angles of the triangle, in which case the rod will be found to be in a line with its original position, but reversed, showing that it has revolved through half a revolution, and that, therefore, the three internal angles amount to 180°.

All these proofs may be presented heuristically, and have real value in opening up new trains of ideas, as well as in consolidating the truths desired to be established.

Many excellent types of examples on this angle-sum property of a triangle may be found in most of the modern Geometries now rapidly appearing. The application to the study and construction, in paper and cardboard, of the regular polygons must not, of course, be omitted.

[This chapter should be studied in close connexion with the next one on 'The Geometry of Position'.]

CHAPTER XII

THE GEOMETRY OF POSITION AND ITS FUTURE CENTRAL RÔLE IN MATHEMATICAL EDUCATION[1]

THE determination of the position of an object relatively to other objects and of the parts of an object relatively to each other—and therefore introducing considerations of shape as a particular instance of position—clearly form fundamental problems in our explanation of the physical world around us.

Round this aspect of geometry may be naturally grouped all other branches of geometry, and it is in the belief that this interpretation of geometrical science will occupy the central position for a long time in future geometrical teaching and research, that the writer devotes such considerable attention to it here. Broadly speaking, the science may be interpreted as co-extensive with geometry itself, in which case we might define geometry itself as the Science of Position. The aim is to suggest lines of development for mathematical education, that will simultaneously fulfil three distinct functions :—(1) form a genuine mental discipline for all pupils ; (2) enable the pupil to apply the subject to future life-purposes ; (3) select and develop the pupil who has the germs of mathematical talent or genius.

The following is not a systematic handling of the subject but a merely suggestive treatment of certain points. Many minds must apply the idea to practical education before its full wealth can develop.

[1] Notes of Lectures to Saturday Morning Class of Teachers; Sunderland, 1903–4. The writer desires here to acknowledge warmly the great advantages he derived about twenty years ago at Edinburgh University from the Lectures of Professor Chrystal and the late Professor Tait.

Maps.

Illustration of how the conventional elementary propositions spring inevitably and naturally from practical problems in Geometry of Position :—

Try to make a map of any simple group of objects : and suppose the map is to be, in size, co-extensive with the distances between the objects : the consideration of shape and drawings to scale develop, easily, subsequently.

The pupils soon find that the elementary figure they must work with is a group of three objects, i.e. a consideration of the properties of triangles is inevitably introduced. [The drawing of a 'map' may spring from a surveying problem in building, from construction in manual work, from the laying out of a garden, from the making of a design, from the cutting out of a dress, &c.]

Analysing the triangle or group of three objects, it is seen that ultimately we must deal with the distance between two objects (which introduces linear measures, sighting, &c.), which again implies the question of the position of one object.

How then am I to fix the position of one object?

[The approximately ultimate aim might be to draw a map of the playground, &c.]

It is seen that position is relative : to fix say one object A, we must already know the positions of—how many others? Two (forming the framework of reference), [e.g. two roads, or intersecting axes : two houses or origins : leading respectively to cartesian (x and y) and radial co-ordinates. If the objects are not all in one plane, then a little discussion of solid figures is necessary, e.g. a sphere in place of a circle, and a parallelopiped or cuboid instead of a parallelogram or rectangle.]

Note that if merely one measurement or datum is given, the position of A is indeterminate. If the framework be one point (say P, whose position is already known) then A may lie anywhere on a locus or curve, called a circle, whose centre is P and radius = the datum (say 2″). In this way the mind becomes familiar with the kind of result that follows when insufficient data are given for the determination of position. And this is a quite natural way to introduce the idea of 'loci', viz. in response to an attempt to solve a practical problem

demanding the use of measured units. (See Table, p. 159: Column VI.)

The locus, of course, is a sphere if the point A may lie in ordinary space (and if only one datum out of the three required for ordinary space is supplied), when the origin or framework is itself a point (P).

Map of three objects :—

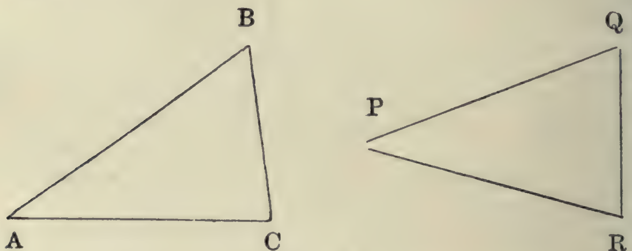


Fig. 44.

Let A, B, C represent the three objects (trees in playground, three pins stuck in blackboard, &c.). [If the pins are removed, how would the pupil replace them in the same relative position ?] A point P to represent A may be placed at our pleasure : Q to represent B may be placed anywhere on a circle whose centre is P and radius $= AB$, say 4 feet. Where is R, to represent C, to be placed ?

The solution of this problem involves a discovery and discussion of many fundamental truths in elementary geometry, and may, in fact, group round itself the substance of Euclid I. 4, 8, and 26, i. e. the determination of the triangle in shape and size.

Clearly it is seen (the procedure being heuristic throughout) that a double application of the already discovered property of a 'circle' solves this new problem.

For R must be somewhere on a circle, centre P and radius $= AC = 3$ feet, say : also it must be somewhere on a circle, centre Q and radius $= BC = 2$ feet, say ; $\therefore$ R lies at intersection of the two circles. Actually construct. The solution is determinate but twofold. The method is a powerful one—called the 'intersection of loci'. With it the pupils should become familiar, using it as an idea comparable in

power with, and complementary to, the use of an actual mechanical instrument.

Then triangle *ABC* is found, by construction and superposition, to be identical with triangle *PQR*, and *PQR* is the required map of *ABC*. Hence the following truth, which a little questioning and discussion will base upon clearly general foundations :—

(A) Two triangles are identically equal (i.e. have the same shape and size), if three sides of the one equal respectively three sides of the other : or, otherwise stated and with equal significance :—

(B) 'Three sides determine a triangle uniquely' (i.e. from three given rods only one kind of triangle can be formed), or, otherwise :—

(C) 'A closed figure (called a triangle) formed by three smoothly jointed rods is a rigid figure.' (See also D below.)

The actual construction of a number of triangles with given data for sides and the use of models of jointed rods soon establish (by appeal to intuition) conviction of the general truth of B. For three rods can be pinned together in one way only, and the figure is felt and seen to be rigid. Whence also follows the form A, for if B or C is true, so must A also be. This point should be, of course, more fully elaborated, though here, especially in the actual handling of bar figures and construction with instruments, the intuitions upon which the above truths are based rapidly develop clearly into consciousness and form the surest support for emotional conviction. A university graduate who had been through six books of Euclid as a boy and had some skill with 'riders', confessed, on handling a triangle of three jointed bars, that not till then did he feel a conviction of the truth that three sides determined a triangle, and see the deeper meaning of Euclid I. 8. This experience happened when he was about 40 years of age.

Let the pupils contrast with the rigidity of a jointed triangle (i. e. a triangle formed of three smoothly-jointed rods) the ease with which a four-bar polygon can be deformed, showing that something additional to the four sides freely jointed is necessary to make a quadrilateral rigid like a triangle. (Illustrate by application to a bicycle-frame where the joints are stiffened by brazing. How many joints is it necessary to stiffen, theoretically ?)

If it is thought desirable, a more formal proof of the above truth may be given (as in I. 8, Philo's proof), but it does not seem really necessary at this stage.

Finally observe that I. 8 or A, B, C above are also really contained in the following statement (which the above construction has established—not, of course, rigorously, but with sufficient rigor relatively to the ages of the pupils) :—

(D) The position of a point (or object) in a plane may be determined by knowing its distances from two other points (or objects)—already given, in that plane.

(Note that if two points are known their distance is thereby known.)

This last form of I. 8 brings into full prominence the central importance of Geometry of Position.

Similar treatment may be given to the substance of Euclid I. 8, I. 26 (as follows in the sequel).

To continue :—

What other ways can you discover of fixing the position of the third point R in the plane in which we are working (and containing P and Q)? or (equivalently, adopting the emphasis on the fixing of a point relatively to a framework) :—

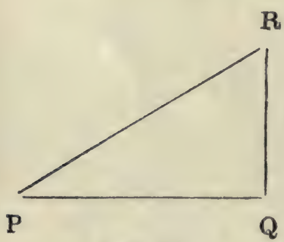


FIG. 45.

What other means have we of describing and determining the position of a point (in a plane)?

(A) If the two angles QPR and PQR be given or known, then clearly the point R can be constructed. [One distance PQ is known, note, implicitly.]

Here again the solution is twofold (in position, of course, only), as the angles can be laid off on either side of PQ.

This solution, therefore, again leads to a similar discussion of the substance of I. 26 and the following truth :—

Two triangles are identically equal if one side is equal in each, and the two angles adjacent to that side respectively equal to the two angles adjacent to the corresponding side in the other, or :—

A triangle constructed with a given side and with two given base angles is unique: or in the two remaining forms as before.

The nature of the evidence is also similar.

This construction also may be naturally developed into the substance of I. 32 (i.e. the angle-sum property of a triangle).

The preceding solutions now inevitably lead to the question: A triangle has six elements, three sides, and three angles. We have seen how to determine it in various ways with three data: are there any other?

Simple Logical Classifications

This leads to a logical classification of the six elements into groups of two, three, and four elements.

The 2-groups are seen to be insufficient. The 4-groups are seen to be over-sufficient (in fact, it is soon found that the giving of 4 data—unrelated—leads to difficulties: interesting developments as to the necessary equations these four must fulfil may be developed subsequently. And in still higher and much later mathematical developments—trigonometrical and analytical—the consideration of the relations between 5-data groups may be usefully discussed).

There remain then the 3-element groups.

The consideration of these, throughout with quantitative as well as qualitative typical figures, leads to the substance of Euclid I. 4, I. 8, I. 26, and I. 32.

Importance of the Triangle.

The central importance of the triangle should be repeatedly emphasized from many standpoints: thus (1) all rectilinear closed figures are divisible into triangles; (2) a triangle is the simplest closed and simplest rigid figure (illustrate and apply here to framework of roofs, bridges, household furniture, &c.); (3) its angles equal two right angles or a straight angle; (4) all triangles and therefore all rectilinear closed figures can be decomposed into right-angled triangles, the most important kind of triangle (hence, basis of mensuration).

The Table.

I now add a brief explanation of some points in the Table (attached).

After a thoughtful study of this Chapter, the teacher will probably find the accompanying Table reasonably self-

explanatory. It epitomizes the fundamental elements in any Science of Position, and, in the hands of an intelligent and competent mathematical teacher, should suggest various systematic lines of development covering a school course of mathematics. One such has already been devised by Mr. Mair (*A School Course of Mathematics.* Clarendon Press, Oxford. 1907).

Models (cardboard, paper, wire, rods, &c.) should be constructed and used throughout.

The Duality of Modern Geometry.

Column II (B) :—Has been partly treated just previously. The teacher is recommended to bring into prominence the duality of Modern Geometry. Thus, in Plane Geometry, a whole system of truths about (straight) lines and points may be doubled in number by simultaneously interchanging point for line and line for point. This follows from the two truths characteristic of plane space :—

(A) Two points uniquely determine a line passing through them (viz. a straight line).

(B) Two straight lines uniquely determine a point lying on both.

For example :—

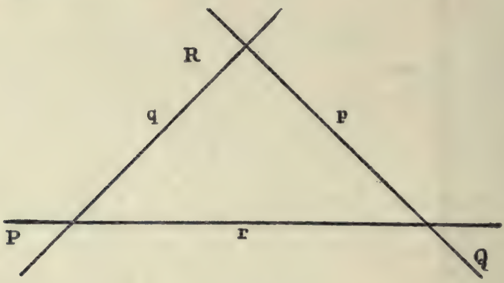


FIG. 46.

A point R may be fixed relatively to two known points P, Q by the intersection of two lines q, p passing respectively through P, Q and making given angles with the line r, determined by PQ.

A line r may be fixed relatively to two known lines p, q, by the joining of two points Q, P, lying respectively on p, q and at given distances from the point R, determined by p, q; and so on.

[If models can be obtained, it may be advisable to develop similar truths in Solid Geometry when it will be found that a point and a plane are now the dual elements, as each

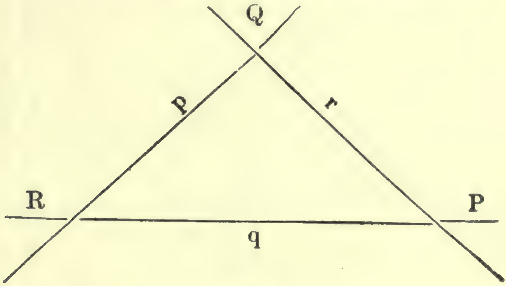


FIG. 47.

requires three data to fix. The fixing of a straight line in space is an excellent exercise, developing simple ideas of cones, cylinders, &c.]

Geography and Astronomy are clearly suggested in connexion with Table, III (*b*) and III (*c*).

Note to Table facing p. 159, Column VI (3) :—

Three measurements are necessary and sufficient to determine the position of a point in space.

This should be shown heuristically both by rectangular co-ordinates (using models, and also walls and floor), by radial co-ordinates, &c. Thus, taking the latter :—

Suppose we have three known points, A, B, C (not less than three will suffice : it is interesting to note how frequently the number three occurs in connexion with our ordinary space), and we wish to fix the position of a point P with respect to these. Let its distance from A be known to be 5 in. : then P must lie somewhere on a sphere of radius 5 in., and centre A. (This also illustrates Table, VI (3), second line.)

Let, further, its distance from B be 4 in. : then P must lie somewhere on a sphere of radius 4 in., and centre B.

Therefore P must lie on the curve of intersection of the

two spheres. This curve (a circle) should be constructed and shown with a model. (This illustrates Table, VI (3), first line.) Moreover, the plane of this circle is perpendicular to the line joining AB.

Finally, if distance from C is 3 in., then P must also lie on sphere of 3 in. with centre C. Therefore the intersection of the previous circle with this sphere determines the required point P. [One may, of course, construct the point, by this method, in three different ways by choosing the three spheres two at a time in three different ways for the original intersection. Many interesting results follow.]

Another excellent method is to fix P by the intersection of two circles (radii 5 in. and 4 in. respectively) round A and B respectively in any given plane containing A and B: then to rotate this plane round AB. Thus we obtain a locus of points (a circle) whose plane is clearly perpendicular to AB. [Models used throughout, if possible. Of course, this implies a simultaneous appeal to the imagination for the understanding of the model, and for the subsequent application of the ideas and figures to more difficult cases.] A repetition of a similar construction between A and C, or between B and C—or the use of a sphere round C to cut the above circle—leads again, in many ways, to the desired result.

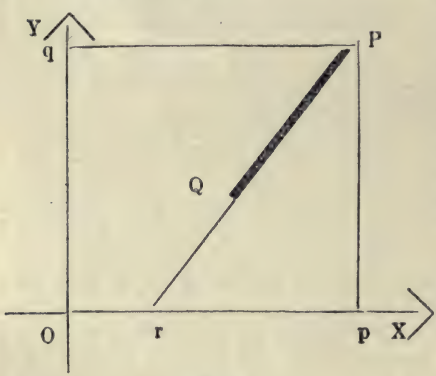


FIG. 48.

It is very important to familiarize the mind with constructions in solids.

Note to Table, II (D): III (d): IV (iv.), &c.:—

Consider a flat piece of cardboard, or a bar (for the problem in hand, the two are equivalent). (Fig. 48.) To fix either of these (say the bar PQ) in the plane of, say, the blackboard (two edges of the blackboard may be selected as framework or axes), first fix one point in the bar, say the end P. This

takes two measurements (viz. two co-ordinates : $Pp = 3$ in., say : $Pq = 4$ in., say : or we may be given two equations connecting the co-odinates x, y of P, e.g. $x+y=7, x-y=1$: or, indeed, equations of higher degree in which case the solution will be multiple instead of unique : in the present brief suggestions, hints are thrown out not merely for elementary teaching but for secondary and higher education generally : the common sense of the teacher will indicate where the line is to be drawn : the main point is that Geometry of Position may form the centre for a systematic development of nearly all branches of mathematics : as a systematic subject it is almost totally neglected).

Once any single point, such as P, is fixed, the rod cannot move away as a whole, but it is still free to rotate round P (in the plane of the blackboard : we suppose the motion confined thereon). To deprive the rod of this final 'degree of freedom' (viz., here, power to rotate) we may fix its inclination to, say, the axis of OX by knowing the angle $PQrX$, say 40°. Thus three measurements clearly suffice to determine the position of the rod.

Moreover, it will be seen, by trial, that no less number of measurements suffices. So that three measurements are both necessary and sufficient.[1]

Degrees of Freedom and Physical Constraint.

Note that the number of 'degrees of freedom' is merely a physical way of expressing the number of independent measurements required to determine the position (Table, IV).

For every measurement required, the body is said to possess one degree of freedom.

The term 'physical constraints' is another physical emphasis of the same aspect, and is obviously complementary to 'degree of freedom'. Every single 'physical constraint' imposed (e.g. by appropriately forcing a point of the body to move along a groove) means one 'measurement' put into operation, and therefore the loss of one 'degree of freedom' (Table facing p. 159, Column V).

Loci.

The three measurements required to determine the position of a plane body (or rod) confined to motion in its own

[1] Some repetition of important truths in a variety of contexts has here and there been deliberately adopted throughout the work.

plane may be specified in an infinite variety of ways, and in this general fact, respecting determination of position, lie an infinitude of suggestions for development of the subject (and for exercises) to the teacher interested in the subject. The specifications may be very simple in nature and fit for children, or so complex as more than to tax the genius of the greatest mathematicians. (It may be added that in few other subjects of mathematics will the teacher find a source of knowledge so interesting both to himself and to his pupils.)

As a simple example :—Suppose that Q is known to be 4 in. from axis OX. Then as P is fixed, Q must lie somewhere on a circle of radius $= PQ$ and centre P. Also Q must lie somewhere on a pair of lines parallel (above and below) to axis of x and 4 in. distant therefrom. Therefore Q must lie on the intersection of this circle with these lines. Where Q is fixed, the rod is fixed. [The solution will sometimes be fourfold.] Interesting cases arise if the circle touches one or both of the lines *r*.

[N.B.—In the above case Q is said to be constrained to lie on a circle *r* : the circle might be, in form, a groove cut in the board, and Q a bead or pin attached rigidly to the rod.]

Again :—Illustration which, mechanically, may be solved by elementary pupils, but which, analytically, demands conic sections :—

Suppose one end P of the rod is constrained to slide along a groove cut along y-axis (i. e. one condition assigned, one degree of freedom lost, one physical constraint imposed)—this condition, analytically, is equivalent to making P's co-ordinates satisfy an equation (a very simple one, here)—while the other end Q is constrained to slide along a groove coinciding with x-axis. Here are two conditions assigned, two degrees of freedom lost, two physical constraints imposed.

The rod can still move as a whole.

[This example shows that though 'translation' and 'rotation' (see Column IV) are convenient terms to describe to young pupils the common and obvious forms of motion, yet, mathematically speaking, translations and rotations cannot always be distinguished : a translation, in fact, is a rotation round an infinitely distant axis.]

What locus will any point on this moving rod describe ?

Try first the middle point of the rod. [This case is sus-

ceptible of geometrical solution by very elementary methods —the locus being a circle. Its discussion leads to valuable and interesting truths (amongst them, substance of Euclid III. 31). The locus should be traced mechanically also.]

Then try any point of the rod, noting the ratio in which it divides the two parts of the rod. (Result, an ellipse. It can easily be traced mechanically. Elementary pupils should be introduced to simple mechanical means such as this, of drawing ellipses—and other conics—and to some of their simpler and fundamental properties.) (It is strongly recommended that a drawing-room and a simple form of laboratory or workroom be developed in connexion with the mathematical and mechanical department.)

To fix the rod completely, one other measurement must be given. This may be specified, of course, in a variety of ways.

Note to Table facing p. 159, Column III (*e*) :—

A Solid Body in Space.

The determination of position of a solid body in space should be carefully discussed.

The following long paragraph might have been inserted as a footnote, and may be omitted without detriment to the main line of thought.

A brief sketch of the essentials of one method only is given: —Fix one point of it, say P. For this three measurements are sufficient and necessary. [This deprives the body of three 'degrees of freedom'.] The body has no longer the power of translation as a whole, but can still rotate—round any one, indeed, of any three axes through the point. For convenience the axes should be chosen mutually perpendicular. This statement—about three axes—is simply another way of stating the theorem (which in elementary teaching it will not be necessary to deal with) of the parallelopiped of angular velocities. Any point on the body now is clearly constrained to move on the surface of some sphere. In particular some second point Q (conveniently chosen : a triangular board is a convenient 'body', and the three vertices convenient points to fix, consecutively as now being described) whose distance from P is known (from the given configuration of the body), say 5 in., must lie somewhere on the surface of a sphere round P as centre and with radius 5 in. Consequently, as Q is somewhere on this

spherical surface, two additional measurements will suffice to fix Q (e.g. its latitude and longitude on this sphere); or, instead of fixing Q relatively to the sphere, and therefore also relatively to P, as the sphere is already fixed relatively to P, and therefore relatively to the original framework of three axes, x, y, z, say, we may fix Q now, by knowing both its distance above, say, the floor, thus giving a plane locus, and also its distance from a wall, giving another plane locus, the intersection of which two planes gives a line cutting the sphere. Having fixed Q, we see now that the body can only rotate round PQ as an axis. Therefore, one more measurement suffices to fix it completely (e.g. any third point R may now be fixed relatively to P and Q, by one additional measurement). This may be assigned in a variety of ways.

In all, then, $3+2+1 = 6$ measurements are sufficient and necessary to determine the position of a rigid body in ordinary space.

Statics and Dynamics.

The Seventh Column will, of course, be undertaken when statics is started (in a properly-equipped mechanical laboratory). It is placed here to show the central position occupied by the subject in hand (Geometry of Position). The elements of mechanics might, with advantage, be begun much earlier than is customary.

Rigid and Deformable Bodies.

Note to Table, I. 4, and remark F :—

This line of inquiry is capable of any degree of expansion thought desirable; appropriately developed (in conjunction with the preceding) it may be made a centre round which it appears possible to group almost any geometrical truths.

The greatest importance is attached to these ideas.

A few examples are given (some elementary: some complex), to indicate, in some faint degree, its wealth of suggestiveness :—

1. The anatomy of the human body.

How many degrees of freedom has the arm? Discuss the nature of its movements.

The upper arm rotates in a ball-and-socket joint. Three measurements necessary to determine its position. Hence three degrees of freedom for this portion.

The lower arm (radius and ulna, of which one revolves round the other), the wrist, and the fingers and thumb, have then to be discussed.

The number of degrees of freedom will equal the sum of the total data required to fix the whole in position. The head, leg, and body generally, if thought desirable, with a model, may be discussed. The movement of the eyeballs is a problem which (with models of the moving muscles) the writer found interesting. Here nature has provided a beautiful and simple mechanical solution of the movements required.

2. Discuss the nature of movement of a common door, of a pair of nut-crackers, of lazy-tongs, &c., &c.

3. How many measurements are sufficient and necessary to fix size, shape, and position (in a plane) of a polygon of 3, 4, 5, 6 . . . N sides?

[All examples, as a rule, as well as being solved generally, should also be illustrated with numerical units and constructed either with models or by drawing-instruments, &c.]

These problems may be attacked in many ways. Taking a quadrilateral :—

Fix each corner. Two measurements for each. $4 \times 2 = 8$. But, if each corner is fixed, the figure is simultaneously fixed in shape, size, and position. Therefore, eight measurements required here.

Or again :—Fix one side, then the adjacent angle, then lay off the second side, then the second angle, then the third side. This fixes the figure in shape and size. That is five data. (Try it.) But, three more are required for position. (See Table, III. d, and p. 162.) That is eight altogether.

And so on for the others.

Note the following general reasoning :—

The position, size, and shape of an N-gon may be clearly fixed by fixing each vertex in turn. This requires $2N$ data or measurements. But three of these are required for position alone.

Therefore, $2N-3$ measurements fix the shape and size of polygon of N sides.

Verify for $N = 3, 4, 5$, &c. How many data for shape alone? $(2N-4.)$

4. Solve the above problems for space of three dimensions.

Here 3 N measurements will determine size, shape, and position, but six (see Table, III. *e*, and p. 165) are required for position alone. Therefore, $3\ N-6 = 2\ (N-3)$ data fix shape and size of a gauche N-gon. Test with $N = 3$, $N = 4$.

5. How many data determine shape and size of a sphere, cube, cuboid, cylinder, cone, tetrahedron, pyramid, dodekahedron, eikosahedron, &c. (Give numerical measurements, and construct.)

How many for shape alone? (One less.)

How many for shape, size, and position? (Six more.)

(This line (viz. (3), (4), (5), above) of development passes over into mensuration.)

e.g.—To fix shape and size of any pyramid on a triangular base, we require 3 data for the triangular base; one datum (the angle) to fix the inclination of any one of the faces to the base (this involves meaning of an angle between two planes, which, by experiment, can be shown to possess a certain obvious minimum property): two additional data to fix the shape of this face (we know its basal line already). But, when the base and one face are determined, it is seen that the whole figure is fixed. Hence $3+1+2 = 6$ measurements, suffice to determine shape and size of a triangular pyramid.

(Reasoning more concisely and generally:—Fix the four vertices. Three measurements required for each: that is twelve altogether. But this fixes not only shape and size but position also. Now six of the 12 data are necessary for position alone. (See Table, Column III. *e*.) One is required for size alone. Therefore, $12-6-1 = 5$ data are required for the shape alone of a triangular pyramid.)

[This last general type of reasoning is applicable to all rectilinear figures (plane or solid). An analogous type of reasoning could be devised applicable to curvilinear figures; but, as the fundamental elements therein would be particular classes of curves, instead of straight lines (the simplest of curves), the reasoning would be more complex.]

Problem:—Specify 6 data for the construction of such a pyramid: have it actually constructed, and calculate its cubic capacity.

6. Discuss data required for shape, size, and position—or shape alone—of circle, square, parallelogram, rhombus, ellipse, parabola, cycloid, &c.

[Cycloid is an interesting curve easily obtained mechanically. Note that all cycloids are similar, as well as circles, parabolas and squares—of the figures mentioned above.]

This type of question, also, develops the theory and practice of mensuration.

7. Apply the theorems of the table to the solution of irregular figures (as in surveying) with given dimensions.

An appropriate co-ordination of (8) (9) and (10) below with mathematics as here outlined should develop the *constructive* abilities of the pupils in a rational, systematic way. There is assuredly a great future in store for this aspect of applied mathematics, now almost wholly neglected.

8. Applications in Engineering and Mechanics.

9. Applications in Manual Work, including wood and metal work for boys and for those girls who have talents in this direction, and domestic economy for girls.

10. Applications in Art (including clay-modelling, &c.).

Intersection of Loci.

11. Construct a triangle, given the base, the opposite angle, and the sum of the sides.

[This is given, like the others, merely as a type of exercise: a very valuable type.]

In problems like this, the method of 'intersection of loci' (falling under head of Table, Column VI) should be developed, and the figure sought should be actually constructed with definite measured units.

The limitations attendant upon the problem should also be discovered. For example, Is the solution possible under the given conditions? Is the solution unique or multiple? Between what extreme limits must the variables lie? What general relations or equations are satisfied by the data given should they be over-specified? &c.

In the present case, only brief outlines are sketched. Note that a triangle is determined in shape and size by three conditions. Let us suppose the three conditions or data here are:—

Base = 4 in.: sum of remaining sides = 7 in.: angle opposite base = 50°.

[Whether a triangle can actually be constructed from these particular data it is left to the pupils to decide.]

The general problem (to be discussed fully, after the class has dealt with particular cases as above) is :—

Base = c, sum of sides (remaining) = k, angle opposite base = a.

What limitations are there to actual numerical values of c, k, and a?

Clearly, if we can fix the three vertices relatively to each other, our problem is solved. Two vertices may be at once fixed, for we know the length of the base. Fixed, of course, relatively to each other : it is not necessary in this problem to introduce co-ordinates and fix the position relatively to a framework, though this method may often be used with advantage, especially if we employ the method of analytical geometry.

Lay down, then, a line AB = 4 in. = base.

We have now to determine the position of the third vertex C.

Apply the idea of Column VI in Table : remove one of the conditions, and C will lie on some locus.

For example, omit the condition that sum of sides (other than base) is 7 in. All we now know is that C is to lie in such a position that the angle subtended at C by AB (i.e. angle ACB) is to be always 50°. Let us assume that the pupils are not yet familiar with the fact that angles in the same segment of a circle are equal : our aim is, in fact, by the presentation of appropriately-chosen problems to stimulate them to a discovery of such properties themselves, under our guidance—such properties being required in order to solve the particular problem in hand.

In the present case, let them by use of protractor, or of a cardboard angle of 50°, actually plot the points where the vertex C must lie. They soon discover that the locus looks like a part of a circle. Let them test if this curve (freely joining the points by a continuous line) is really a circle, so far as their instruments will allow.

The problem in hand now is :—

How are we to tell whether a given curve is a circle or part of a circle?

If this problem (the substance of Euclid III. 1) has not previously been solved, then it must be carefully discussed here, and the pupils be given to understand that until this new problem is solved they cannot hope to solve the original.

This mode of procedure has many and great advantages: it excites genuine and lasting interest, stimulates originality, initiates the pupil gradually into an understanding of scientific method and procedure, and correlates the parts of mathematics together in a rational, systematic, and impressive manner, so that only those truths are made prominent which occupy a really useful position in the development of the subject.

Some Historical Aspects.

Moreover, broadly speaking, it may be claimed with reasonable truth that, whatever be the practical problems the teacher starts with originally, a thoughtful and thorough treatment of them will necessarily group together the main truths of the science ere the end be reached. The method has full historical warrant, from our principle of parallelism between race and individual. It was in the attempt to solve certain definite problems of this nature that the whole systematic Greek Geometry became developed. Though some of those problems were never solved till modern times, yet the very attempt led to the greatest and most important discoveries.

The famous problems solved by the Greeks were :—

1. To construct a square equal to any given plane polygon.
2. To construct the regular solids. (All, of course, with the limitations of instruments to straight-edge and compasses.)

The first problem is fully solved by the time one reaches Euclid II. 14 : the solution of the second is contained in the later books.

The famous problems not solved were :—

The squaring of the circle.

The duplication of the cube (the Delian problem).

The trisection of any angle.

It has been shown (in the case of the first and third, only as recently as the last half of the last century) now that, with the limitations of instruments imposed by the Greeks, these three problems cannot be solved, but it took considerably over two thousand years to do this.

Historical points like this should be given to our pupils directly they can understand them.

The above is merely a brief sketch : the teacher is referred to any history of mathematics for details.

Circles and Ellipses.

Returning to our problem (see page 170) :—

This is a convenient point, evidently, for setting about a discussion of the fundamental properties of circles : hence —Fundamental Properties of the Circle.

(This, once started, may be carried as far as one pleases : the rational point at which to stop is when one has discovered sufficient properties to solve the original practical problem which gave rise to the discussion and consequent development.)

The interesting problem at the foot of page 170 will involve the development of the substance of Euclid III. 1, 3.

Let us, then, assume that it has been found that the freehand curve passing through the points found by experimental construction as suitable for the vertex C, and therefore forming the locus of C, is, as nearly as we can tell, a circle. If so, then we have the suggestion, due to our construction, that the angle in a given segment of a circle is constant (i. e. of fixed size).

How can we be sure of the general truth of this property ?

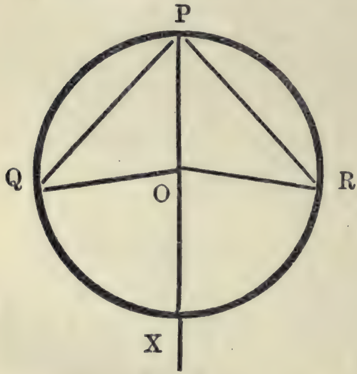


FIG. 49.

A general proof is easy on the lines of Euclid. Note how many general proofs of fundamental theorems in Euclid can be given without a long chain of preliminary theorems, and merely by simple and direct methods, provided we are willing, at frequent points, to make certain intuitional assumptions which every pupil, with sufficient experience, is ready to admit, and feels the force of. In the present case let us examine Euclid's proof and see what it implies :—

The essence of the evidence is that (O being centre of circle, and taking one of the typical cases) :—(Fig. 49.)

Angle QPR = angle QPO + angle RPO = $\frac{1}{2}$ (angle QPO

+ angle OQP) + $\frac{1}{2}$ (angle RPO + angle ORP) = $\frac{1}{2}$ angle $QOX + \frac{1}{2}$ angle ROX = $\frac{1}{2}$ angle QOR = $\frac{1}{2}$ of a fixed angle.

All this involves merely :—

1. Angles at base of isosceles triangle are equal. (A truth proceeding from intuitional ideas of symmetry—of course, it should be tested.)

2. The corollary to the 'angle-sum' property of a triangle (previously proved).

[N.B.—The actual procedure in teaching should be, it need hardly be said, very different from the above. Briefly sketched, the following is suggested :—

Draw figures and measure the various angles involved, tabulate these, and ask for relations between the angle at centre and angle at circumference.

Soon the whole general essence of the evidence will be seen, and should be stated in clear and definite language.]

After this property of a circle has been established, another (especially as it is required for a full discussion of the constructive triangular problem started with) should at once be developed ; viz. the fact that in a cyclic quadrilateral the sum of opposite angles equals two right angles.

Treatment as before. Draw actual figures and tabulate angles. Then get proof out of pupils by simple application twice of the just-discovered fact that angle at centre is double of corresponding angle at circumference. [The total angle at centre being four right angles, the general proof is obvious.]

The converse of these propositions should equally firmly be impressed, viz. :—If in a quadrilateral a pair of opposite angles amount to two right angles, then that quadrilateral is concyclic : or (equivalently in substance) thus :—'If three points are taken at random, can a circle be put round them ? Specify three points, and ask for a solution with the minimum number of operations with the straight-edge and compasses. What exception is there ? Explain this. Can a circle be put round any four points (in a plane) taken at random ? Try. If not, why not ? What conditions must these four fulfil, if it is possible ?'

(The pupils should also examine the extreme case when the third vertex coincides with an extremity of the base, leading to tangents and III. 32, and its practical application to the drawing of tangents to circles from external points.)

Returning again to our original problem (page 169, No. 11) we have now a means of constructing the locus of the vertex of the triangle, where the vertical angle is given and also the base subtending it.

Next :—Let us omit the condition that angle at vertex is 50°. We have now the problem :—

What is the path (locus) of the vertex of a triangle, the sum of whose sides is given, and also its base ?

Here, sum of two sides = 7 in. : base = 4 in.

As before, let pupils construct by trial and error the path of the vertex. It is an ellipse (of which the two base corners of the triangle are the two foci). A simple mechanical method of drawing ellipses may be given founded on the above definition (that the ellipse is the locus of a point moving in a plane, so that the sum of its distances from two fixed points in the plane is constant), viz. :—Fix two pins at the two required foci (A and B, say: in above case extremities at the base of the triangle). Tie one end of a thread, 7 in. long, to A-pin, and the other end to B-pin. Then a pencil-point inserted (see Fig. 50) at P, and moving so as to keep the thread taut will trace out an ellipse. What length has the major axis, with respect to length of thread ? (Major axis is axis through AB, and terminated by elliptical circumference.)

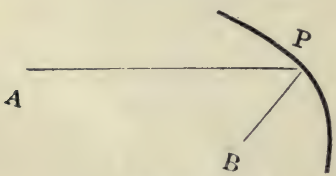


FIG. 50.

The relation of this mechanical method to that given on page 164 is of interest, and leads to important developments.

A few simple properties of an ellipse may be given.

It may be utilized (with obvious limitations) to find the path of a ray of light passing from a source A and reflected by a plane mirror through a point B.

[That the path APB, when P is the point of reflection from the mirror, is the minimum may be shown, by simple methods, to be when angle of incidence = angle of reflection : and the introduction of the ellipse, with the above definition and mode of construction, should be fairly obvious, leading to an additional property (a tangential) of the ellipse, that

the tangent at any point makes equal angles with the two radii vectores at that point, viz. AP and BP.]

Although this problem can be solved more simply, yet the teacher will observe that many problems are suggested which it would be impossible to solve with the conventional Euclidian instruments (straight-edge and circle). At the same time this procedure throws into prominence the kind of problems which can be solved by use of these two simplest geometrical instruments, and a clear distinction must be always maintained on this point. Indeed, one of the practical aims of geometrical study is to attain skill in constructing figures with the least number of operations, in the use of ruler and compasses. (The production of a straight line itself by link-motion (Peaucellier, Hirst, or other) is valuable, and the history of Watt's 'parallel-motion' may be introduced herewith.)

Having discovered how to construct the two loci obtained when one condition is neglected, we may now solve the original problem by the intersection of these two loci: the solutions will be given by the intersection of the circle and ellipse. Construct them.

Of course there is a method, in this problem, of constructing the triangle without use of an ellipse, by aid of two circles (pupils will note, however, that a circle is an ellipse whose two foci are identical) thus keeping it within the bounds of the Euclidian Geometry.

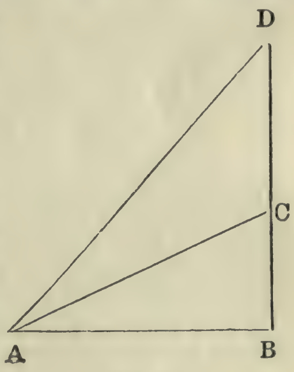


FIG. 51.

The discussion of this method also leads to discovery of the above-mentioned properties of a circle.

Type-Figures and Scale-Figures.

Fig. 51 (also see p. 177) is a type-figure—a type-figure being a figure not drawn to scale, but simply drawn freehand, for purposes of discussing the geometrical properties of the figure. Such are Euclid's figures. It would be a grave blun-

der to suppose that we can dispense with this kind of figure and always draw figures to scale. This very example proves, indeed, the absurdity of such an idea—for how can we draw the figure to scale here (without mere guesswork) when we do not know the length of the sides. It was a great weakness in the use of Euclidian Geometry for schools that no figures were drawn to scale, all being mere typical (or symbolical or standard) figures for general reasoning; but it would, in my opinion, be a still greater blunder to go to the other extreme and dispense with such figures entirely. Both kinds are necessary for genuine progress and rational mastery of the subject. It will be noted that the figure drawn to scale is predominantly useful in the experimental aspects, while the freehand types are predominantly useful in the establishment of general truths (as an instance of the latter—the use of type-figures in formal general reasoning—note that a type-figure is necessary in arguing backwards from the result (or construction) required to the method of actually obtaining it, as in the present problem). There is much danger nowadays, in the eagerness to reform, of overshooting the mark and losing the really valuable and indispensable elements characteristic of the old Euclidian teaching at its best.

Configuration.

Moreover, the teacher and pupil should develop together the conception, inherent in position, of configuration. Every figure possesses, in addition to position as a whole, the three aspects of size, shape, and configuration. Thus, all triangles have the same configuration, though they may vary widely in size and shape: identity of configuration is that which, indeed, distinguishes the triangle from other species of figures. No attempt here is made to define precisely 'configuration' otherwise than by implication. It implies the general spacial constitution of a figure. It is not difficult to see that a necessary, but in general insufficient, condition for sameness of configuration of two figures is that the number of data required to determine them in size, shape, and position be the same. Moreover, even this general statement should be limited to rectilinearly-constructed figures (either in plano, or in solido). Thus the figures on pages 175 and 188 have the same configuration, however they may differ in size and

shape. Of these three aspects—configuration, size, and shape—configuration is the essential element in a type-figure, while size and shape are the essential elements in a scale-figure.

Of course a knowledge of the size of every element of a figure would determine also the precise shape provided the configuration were given too. (For the configuration tells us how to fit the elements of the figure together.)

It is essentially in discovering the properties of new spacial configurations that Euclidian or any other branch of Geometry develops, and in doing so it makes continual intuitional assumptions over and above the initial axioms or postulates.

In the above type-figure (51), if ACB were 50° and DB = 7 in., and ACB the triangle desired, then clearly $AC = CD$ $\therefore$ angle CAD = angle CDA = $\frac{1}{2}$ angle ACB = 25°. Therefore D lies on a segment of a circle with AB as base and containing an angle of 25°. Hence the intersection of this circle with the circle round B as centre and with a radius of 7 in. will solve the problem, if the problem is soluble with the given measurements.

Union of Theory and Practice.

Do not fail to get your pupils actually to construct such circles, choosing obtuse as well as acute and right angles. The substance of Euclid III. 33 is involved and III. 32 as particular case of III. 21. It is astonishing to find how many lads can write out this proposition and even work riders upon it, but cannot, with instruments, actually apply it to a numerical construction. Experience shows that neither the capacity to do theoretical logical 'riders' (as one school supposes) nor the capacity to construct figures in practice with the instruments (as equally mistakenly supposed by the opposite school of teachers) is by itself alone sufficient evidence that there is rational mastery of the study. The capacity to do both must be present—theory and practice together and in harmony, and neither developed at the expense of the other.

Finally: in connexion with this problem (page 170), (trying all varieties of loci involved in the case), if we neglect the condition that AB the base is 4 in., we have the problem :—

What is the envelope of the base of a triangle when the

opposite angle is fixed in position and size, and the sum of the two enclosing sides given?

The figure may be constructed by trial and error: lay off an angle of 50°, say O: then choose P and Q, so that $OP + OQ = 7$ in.: join PQ: do this in a great many ways, and so that the successive positions of PQ are always close to each other. The varying line PQ will always touch a certain curve, which is called the 'envelope' of PQ. What species of conic section is this envelope?

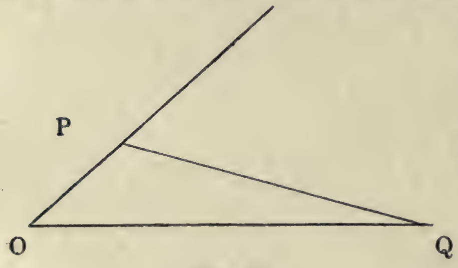


FIG. 52.

The conic is a parabola.

The name may be introduced, and the generation of the various conics, with continuous passage from one into another, shown by models of sections of a double cone, including the point, the circle, ellipse, parabola, a pair of coincident and a pair of parallel straight lines, the hyperbola, a pair of intersecting straight lines. It is not suggested that 'Conic Sections' should be, as at present, treated as a set study, but that they should receive some such treatment as is outlined in the spirit of this chapter. The writer thinks that geometrical 'Conic Sections' as a set study (or isolated from the analytical aspect) will disappear.

Scientific, Intuitional, and Experimental Constructions.

Just as the nature of the evidence for geometrical truth may be broadly classified into experimental, intuitional, and scientific, so may the actual modes of procedure for the construction of figures, with given specifications or data, be classed under these three divisions. And it will be found that, in general, the general characteristics of these three ideas underlying evidence or proof also broadly hold of the same ideas underlying construction. To illustrate :—

Suppose the teacher draws a circle on the board, destroys all traces of the centre, and asks a pupil to find that centre.

The three Types of Constructions.

There are (broadly speaking) only three distinct ways of proceeding :—

I. Experimental Construction. By trial and error, guesswork.

One may place the extremity of one leg of the compasses at different points and try various openings of the compasses until one has hit upon the right radius and the right centre. The number of uses of the instrument, technically called an 'operation', is a matter of luck in such a procedure. Of course the number may be greater or less according to the intelligence with which the operator guesses.

II. Intuitional Construction :—

Take a sheet of transparent paper : lay it over the circle on the blackboard : prick out the outline of the circle on the paper : cut out the outline : we have now a circular piece of paper. Fold it into two semicircles covering each other (this can be done quite systematically, without any guesswork, by an obvious procedure), thereby making a crease which must be a diameter. Repeat the operation to obtain another diameter, or, as simply, bisect the diameter itself by folding it in two halves. Thus is the centre of the paper circle obtained. Reapply to board and prick through centre.

The procedure here is quite systematic, observe, though somewhat lengthy. For it involves only a definite number of operations, which is the same for every one using this method, and which would be the same to whatever circle the procedure were applied. The experimental method above in contrast is unsystematic, involves a quite indefinite number of operations, and is a method which, applied to different circles, would in general require a different number of guesses or trials. At the same time, for entire circles within a familiar and reasonable size, the eye may by practice attain such skill that, practically speaking, the experimental method is the quickest of any. But many limitations to its use are obvious : e.g. suppose only a small portion of the circle is given : suppose the circle is too big, &c. : while in either case an intuitional method could be discovered of dealing with this new kind of problem if the above failed. For we may use any in-

strument or procedure we please. Now consider the rigorously scientific procedure.

III. Scientific Construction (of the centre) :—

Here we are confined to the use of certain definite instruments. In this case, straight-edge and compasses. The aim is, with these, to solve the problem with the minimum number of operations possible. Note the analogy between the practical aspect of the scientific solution of the problem and the theoretical aspect of the ideal (scientific) proof. In each case with a limited number of data (postulates or instruments respectively), we aim at the solution employing the least number of steps (reasons in proof or operations with instruments, respectively). From this analogy we begin to see the vast importance of scientific method. For it enables us, in the practical application of the ideal to the real, to effect the maximum production with the simplest tools or instruments with the minimum amount of work (operations). This is one of the ultimate and all-sufficient justifications of the eternal human striving towards the scientific ideal in the proof, development, and application of truth.

In the present simple problem we have merely to select three convenient points on the circumference—say P, Q, R. With P as centre and PR as radius, describe a circle (one operation). With Q as centre and PR as radius, describe a circle cutting the former in A and B (second operation). With R as centre and PR as radius, describe a circle, cutting the first circle in C and D (third operation). Pass a straight line through AB (fourth operation). Pass another through CD (fifth operation). The intersection of the two straight lines gives the centre desired. In all, five operations (Euclidian).[1]

The question springing naturally from this problem (see also Table, p. 159, Column VIII) is :—Can a circle be drawn round any four (uniplanar) points chosen at random? The discussion of this problem leads to the discovery and development of the substance of Euclid III. 21, 22, 35, 36, 37, and converses of each, and Euclid VI. D (Ptolemy's Theorem). These, in addition to general treatment, should of course receive also numerical illustration : e.g. construct the circle passing through the four points P, Q, R, S, where

[1] Let the pupil discover what limitation must be imposed on the choice of the positions of P, Q, R if the construction is to be practically effective and theoretically correct.

PQ, RS are any pair of lines intersecting at O such that $PO = 4''$, $OQ = 3''$, $RO = 2''$, $OS = 6''$. A very simple and elegant proof of Euclid III. 35 and 36 may be obtained from the simple property as to proportionality of sides, in equiangular triangles—developing previously the properties of similar figures by Book I methods, under the assumption that the magnitudes used are commensurable.

Economy in the Use of Instruments and Reasoning.

Boys will not, in finding the centre of a given arc of a circle, by any means always hit on the minimum number of operations on the first trial. Many will use four points, to start with. This economy of science in the use of instruments and reasoning is a point which youths will be found to appreciate greatly. It may be held before them as a stimulus urging forward to the scientific grasp of knowledge. Interesting and valuable alike is it to count the number of operations required from the beginning in the Euclidian solution of many problems; e.g. How many times does Euclid use ruler and compasses to make a square equal to a given rectilinear figure—one, say, with five sides, and with given dimensions? Very few trials of this kind, combined with the attempt to carry them out with quantitative accuracy, soon suffice to convince the teacher, and pupil too, that the Euclidian constructions are not, in general, practically realizable. Hence the need for more modern and rapid geometrical and yet equally scientific methods, such, for instance, as developed by Steiner, Cremona, &c. It must be confessed, however, that many of the so-called modern methods were really known to Apollonius and other Greek mathematicians—if not to Euclid. It will be generally found that the Euclidian construction in actual realization with instruments, from the beginning (as is necessary in a practical construction), is so complex, so frequently requires more space for its realization than the paper or board will allow, and so inexact in the actual result (owing to the number and kind of operations demanded) that the Euclidian method, in details, soon has to be given up. Some of the modern methods (based, of course, on Euclid, but proceeding beyond him) should be studied by teachers even of Elementary Geometry.

For practical purposes, a combination of trial and error (experimental—developing skill), of ingenious devices (in-

tuition—developing imagination and originality), and of systematic conventional procedure (scientific—developing thoroughness, order, and system) is shown by long experience to be the best. The application of this fact to teaching is obvious, without further comment.

Historically :—The experimental procedure in construction was developed first, then the intuitional, and finally the scientific. The Egyptian Geometry is mainly experimental, the Hindoo and the early Greek mainly intuitional, and the later Greek mainly scientific.

Displacement of a plane figure in its own plane.

See Table, p. 159: Column IV (iv) and Column VI (4), &c. :—

A most valuable, instructive, and interesting set of truths may be developed in connexion with a certain fundamental theorem in coplanar (or uniplanar) displacement of a rigid figure (i.e. displacement or motion which takes place in one plane). A line of attack is briefly sketched in the following description.

The truth to be ultimately established is that 'if a plane rigid figure be displaced in any way in its own plane, there is always (with one exception) one point of it common to any two positions ; that is, it may be moved from any one position to any other by rotation in its own plane about one point held fixed.'

(A) By Table, Column III (*d*), it is seen that to fix the position of a rigid plane figure on the blackboard three measurements are necessary and sufficient.

(B) One or two experiments suffice to show that knowledge of the position of a simple rod rigidly fixed to the figure will enable us to fix the position of the figure itself (e.g. the rod may be a given side of a triangle : after the rod is fixed, the triangle can clearly be built up on the rod, as we are supposed to know its shape).

(C) In our experiments, then, in displacement, it is easier to concentrate attention on the motion of a simple rod.

It is clear that the figure cannot occupy reverse positions as the triangles ABC and $A'B'C'$ in Fig. 53. To get from one of these to the other we should require to lift the triangle right out of the plane, turn it over, and then put it back again : but we are discussing only displacements possible in the plane. This difficulty really involves, let it

be noted in passing, an assumption in Euclid I. 4. The difficulty in establishing congruency of reverse or opposed figures became acute in Greek Geometry, in solid mensuration, when the pyramid was reached. Here, there is now no *deus ex machina* (space of higher dimensions—here four) to overcome the difficulty, and the only alternatives appeared to be either to make an (intuitional) assumption that two reverse pyramids all of whose internal dimensions are identically equal must be equal also in capacity, or to have recourse to infinitesimals—which last the Greeks actually did, in essence. This curious but common relation of two otherwise identical figures should be pointed out: e.g. a pair of gloves: right and left hand: object and image in mirror, &c.

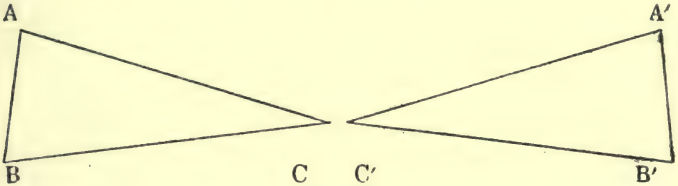


FIG. 53.

Now, what kind of movements can a rod have (in a plane)? See D, E, F, following.

(D) *Translation*: in which any two positions form parallel and equal (straight) lines.

Parallelograms and their properties may be profitably further discussed here, if they have not already been dealt with. For any two positions of a rod in simple translation form two opposite sides of a parallelogram.

The formation of figures by movement was discussed by the Greeks, though the ideas have been neglected in our modern editions.

Many excellent exercises and ideas can be developed from this one simple and familiar idea. Note how geometry thus becomes associated with the concrete world.

Practical illustrations should be discussed, and actual experiments made.

The idea of 'shear' (change of form without change of volume)—as in the flow of a river—can easily be explained here.

The mensuration of a parallelogram will also be suitable

here, if not already considered (the fundamental truths dealing with the areal equality of parallelograms may advantageously be based upon translation, displacements, and actual superposition).

What measurements are necessary to describe the change of position (in translation) ?

(E) *Rotation* : a movement in which each point describes a circle or part of a circle.

First fix one end, say Q. Revolve the rod round Q. Q is the 'centre of rotation.' Find out the properties or character of this movement.

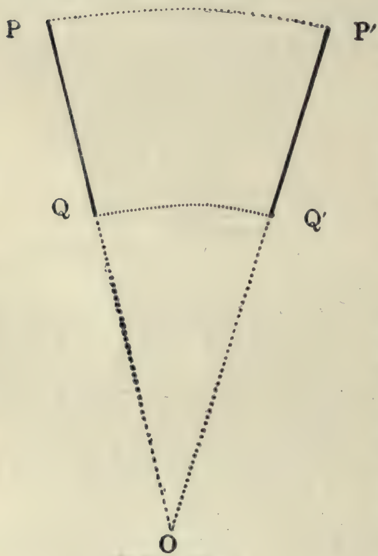


FIG. 54.

How many and what measurements fix the amount of change of position (assuming Q already fixed) ?

One ; angle between first and last position.

Here the properties of a circle dealing with arcs and angles at centre and circumference can be appropriately and simply discussed : e. g. that equal arcs subtend equal angles, &c. : viz. substance of Euclid III. 26, 27, 28, 29, and of VI. 33.

Of course, in the treatment of the latter (Euclid VI), magnitudes will be assumed commensurable, as all practically constructed and measured magnitudes are. Only ideal magnitudes possess the property of incommensurability : and these at present may be neglected.

A slightly more complex form of experiment is to imagine the rod (looking at a type-figure) rotated from PQ to $P'Q'$ about O ; O being a point not on the rod, but, of course, in line with the rod in either position. (See Fig. 54.)

Also PQ should be rotated round a point between its ends,

and results noted—as regards both theoretical conclusions and practical uses.

Finally, a triangle, or other rigid figure, should be rotated so as to familiarize the eye and hand with more complicated forms.

A carpenter's two-foot measure, with stiff joints, opened out and bent as in Fig. 55, makes an excellent figure here to experiment with.

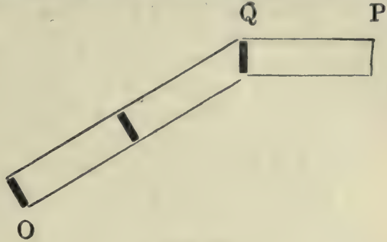


FIG. 55.

Further: to describe the total change of position, if the centre of rotation is not known, we clearly need three measurements, viz. two co-ordinates of centre of rotation O, and one angle, X, to measure the amount of rotation round O.

(F) *Combined Translation and Rotation* :—

Take any two positions of the rod PQ in the plane: say PQ and $P'Q'$. To get $P'Q'$ from PQ we might (1) translate PQ till Q coincides with Q' (i.e. move the rod parallel to its original position PQ) and then (2) revolve the rod in its new position round Q' until it coincides with $Q'P'$. Now, for (1) two measurements are necessary (viz. co-ordinates of Q'), and for (2) one measurement (viz. angle of rotation). Or, we can, of course, first rotate and then translate. Therefore, three measurements suffice to construct the new position. It therefore looks possible to get PQ into the position $P'Q'$ by a mere rotation, because for a general description of a rotation (see E above) three measurements were found necessary. How can we do this?

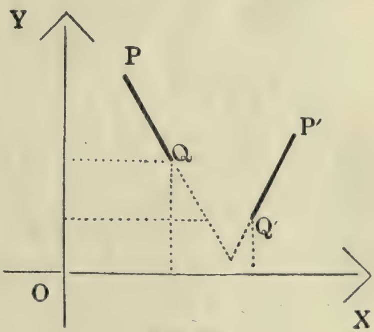


FIG. 56.

(G) *Centre of Rotation* :—

The problem is : can we get the rod from PQ to $P'Q'$ by one movement of rotation ?

Let the pupils actually try this problem by 'trial and error' method. They may fail, but a persevering attempt will teach them much, and prepare for the solution.

First the pupils will note that any two known points on it fix the rod in position (e.g. the two extremities). Hence we are really concerned not at present with the whole rod but with merely two points on it. Now, examine the successive positions of the two extremities. P moves to P'. Moreover, by assumption P is to move to P' by rotation, hence P is to traverse a circular arc.

The question is : Can we decide upon what circular arc P moved ? Discussion now turns on the rotation of one point about another. Clearly (by trial and error) we soon find that P might have got to P' in an infinitude of ways by moving along any one of a number of arcs of circles whose centres all lie on the perpendicular bisector of PP'. The geometry of this is very elementary, and gives a very significant meaning to the theorem that the locus of points equidistant from two fixed points is the perpendicular bisector of the line joining them, in uniplanar geometry. I have sometimes found it instructive to draw attention to the solid geometry solution and to construct it, viz. the plane perpendicular to and bisecting the line PP'.

Hence the centres of rotation by which the extremity of the rod must have gone from P to P' all lie on the line bisecting PP' at right angles. Note that these experiments clear up ideas, usually very hazy, about the rotation of points round points as distinct from the rotation of a finite figure like a rod.

The very same solution shows that the other extremity of the rod Q must have got to Q' by rotating round one centre on a line of centres lying in the perpendicular bisector of QQ'. Where, then, is the single centre we need for both movements ?

Clearly, then, the centre of rotation common to the rotation of both extremities, and therefore, of all the points on the rod, is the point O where these two perpendicular bisectors intersect. [The exceptional case originally mentioned occurs if they do not intersect. In this case they are

parallel, and the displacement turns out to be a simple translation, or, if one prefers, a rotation round a point at infinity.] This point O, therefore (the intersection of the loci) may now be constructed, and the rod can be actually brought from one position to the other by rotation round this point. [It is convenient to familiarize the pupils with the idea of rotation round an axis passing through this point and perpendicular to the plane—for future developments.] This should actually be done :—

1. For a rod.
2. For a triangle.
3. For any irregular plane figure to which a rod is rigidly attached.

Moreover, exercises involving measured units should be introduced. Many new ideas will open out.

Clearly, if we know the successive positions of two points of the figure, we can construct the centre of rotation.

If the two new points are two succeeding points traversed in continuous motion by a moving figure (or body), then the problem assumes, in general, the form of drawing a tangent to the curves traced out by the two points of the body, and we get a centre of rotation 'instantaneously' valid only.

If thought advisable, this idea may be illustrated by and applied to a rolling wheel, when the instantaneous centre of rotation is the point of the wheel in contact with the ground. The discussion of 'centrodes' in engineering courses in later years at college will profit much by this kind of preliminary elementary training at school.

An elegant application of the above ideas may be made to re-prove the fundamental congruence theorems of a triangle by actual displacements of one triangle so as to fit the other. The sign $\simeq$ is convenient for congruence; the sign $=$ indicating equality of size and $\sim$ identity of shape. Thus if a triangle ABC is congruent with $A'B'C'$ (i.e. equal to it in every respect) we denote this relation by the equation, $ABC \simeq A'B'C'$. (The sign $\#$ for 'parallel and equal' may also be recommended here.)

On this point the teacher familiar with German may be recommended Mahler, *Ebene Geometrie* (*mit zweifarbigen Figuren*); Leipzig, Göschen'sche Verlagshandlung—costing about 10*d*.

For teachers knowing little or no German, the reading of

a German book on some technical subject, such as Geometry, with which they are familiar, makes an excellent introduction to the language .

Application to Trigonometry :—

Any right-angled triangle is a two-data figure, i.e. two measurements fix its shape and size. One measurement suffices for shape alone. This may be an angle, or the ratio of any two sides, or, &c. Hence arise the trigonometrical ratios and trigonometrical tables.

This subject should be developed in close connexion with a simple theory of similar (commensurable) magnitudes, and tables of the Sin, Cos, Tan, &c., constructed from measurements made by the pupils (corrected by printed tables).

The application of these tables to surveying (or the problems developed in surveying) should be preceded by a discussion of the determination of the figure, and the

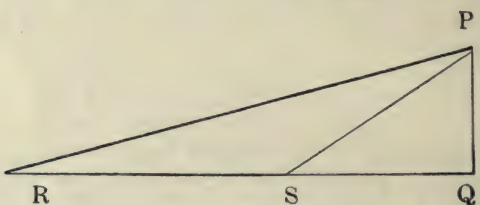


FIG. 57.

calculations tested by constructions made to scale.

As an example, take the problem :—

Find the height of a chimney which is such that, on walking towards it 100 feet in a horizontal line through its base, the angular elevation of its top changes from 30° to 45°.

The figure (in shape) is on trial clearly seen to be constructible as a three-data figure. This is, in fact, the precise number of data given—90°, 30°, and 45°. A figure should now be constructed with the shape required.

Start with $PQR = 90°$, where PQ is the chimney, Fig. 57. S is fixed by making $QPS = PSQ = 45°$. R is fixed by making $SPR = 45° - 30° = 15°$.

Now, we know one actual length, viz. $RS = 100$ feet. Therefore the others, including the height of chimney, can

be easily got from measuring the figure, which has clearly been constructed to scale, for it is of the shape required.

Subsequently the value of PQ should be calculated (with help of formulae and tables) in the usual trigonometrical way, and the results compared. Sometimes the reverse order is valuable—the calculation first and the verification by construction to scale afterwards.

Connexion with Analytical Geometry (and its elements, simple graphs) :—

This connexion needs only to be mentioned in order to be at once grasped. Clearly the very essence of Analytical Geometry is the investigation of loci. Not only the elements of this subject (simple graphs and the use of squared paper) but the higher branches also would vastly benefit by being developed in close touch with and as a natural consequence of the ideas underlying Geometry of Position.

Connexion with Geography :—

This has been emphasized already. However, the general deficiency of people in the power of orienting themselves and their surroundings—particularly, of course, is this so with the blind—is a fact which teachers of Mathematics and Geography should consider the causes of : this power of orientation would doubtless not be so rare in adults, if the fundamental ideas in the Geometry of Position were systematically developed in childhood and in youth.

General Remarks :—

It will have been noted by any teacher who has studied this Chapter carefully that in introducing certain fundamental ideas from Geometry of Position we have inevitably and naturally been forced to develop practically all the really important truths contained in the Third Book of Euclid in order to solve and understand the solution of a few practical problems confronting the pupil at the oustet. The same set of fundamental truths about a circle may spring from any one of an unlimited variety of problems : so that the teacher's choice is endless : the spirit only it is that is here advocated.

It is hoped that solid evidence has now been given to show that the subject in hand is destined to play a central part in future Mathematical Education, and, one might add, a vastly greater rôle than it has yet done in Mathematical scientific development itself.

Algebra.

Although Algebra [1] has not been explicitly mentioned in the above sketch, the teacher will find that practically all the Algebra required for general education can be naturally brought into the scheme proposed.

Historical Note :—

The descriptive title ' Geometry of Position ' was used at least as early as 1803 by the celebrated Carnot (father of the equally famous Physicist, Sadi Carnot) in his *Géométrie de Position* : it was transferred to Germany under the name of *Geometrie der Lage* : its Latin title is *Analysis Situs*. Though these four names (with others which might be added, e. g. *Descriptive Geometry*, *Topology*, &c.) have not by any means always covered the same ground owing to the absolutely unlimited number of branches into which the subject is capable of developing, yet there is sufficiency of fundamental idea common to all to justify the similarity of the titles.

[1] For an example of a systematic course, designed for secondary schools and largely applying these principles and methods, I would refer the teacher to Mr. David Mair's *School Course of Mathematics* (Clarendon Press, Oxford). See also Dr. Percy Nunn, *The Teaching of Algebra* (Longmans, Green & Co. : Modern Mathematical Series).

CHAPTER XIII

PRELIMINARY LESSONS ON SIMILAR FIGURES[1]

I WANT you to try a little experiment. First of all draw figures like these :—

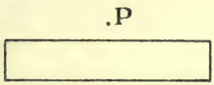


FIG. 58.

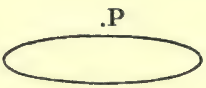


FIG. 59.

a rectangle, which is very long and narrow, and then an oval, also very long and narrow, the figure on the right. Now take a point P, pretty much in the position I have placed it in, outside each figure. Then I want you each to draw through P a figure round the rectangle *of the same shape.* Also a figure through P round the oval and *of the same shape* as the oval. [Interval of a few minutes here.]

I find that nearly all of you have got it wrong. This is what you have drawn :—

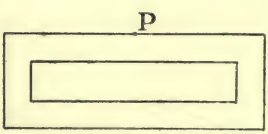


FIG. 60.

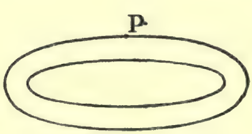


FIG. 61.

I will leave these on the blackboard and we will come back to them subsequently and find out exactly where they are wrong.

What would you take as the test of similarity ? When would you say that two figures are of the same shape ?

[1] To twelve students in an evening class: (shorthand notes). The students have read *four* books of Euclid or their equivalent (at another school), and some a little of the sixth book.

ANSWER: When the sides and angles are in the same proportion.

I see you have at some time or other learned the definition of similar figures—though imperfectly. Will those hold up their hands who have, at some time or other, learned this definition? The words you have given me do not appear to have any very clear meaning in them. What do you mean, for instance, by the angles being proportional?

ANSWER: I suppose it must be when the angles are the same.

That is quite a different statement. Is it then sufficient if the angles are the same and all the sides are proportional? I also want to know which angles and which sides?

ANSWER: When the *corresponding* sides are proportional.

Now we are coming to the root of the matter. '*Corresponding* sides'—that means in the first place that, if two figures are similar, there must be a *correspondence* of some kind between the two. There must be something in one corresponding to something in the other. Will some one put this idea more definitely for me?

No answer.

There must be a correspondence—what sort of correspondence?—between the two figures if they are similar. What kinds of elements are all figures in mathematics made up of?

ANSWER: Lines.

Anything else? Observe that all figures are made up of points, lines, and surfaces; but, since we are here dealing with *plane* geometrical figures, for our present purpose, figures are made up of—What?

ANSWER: Lines and points.

Here is a simple figure, consisting of one line, a, and one point, P. If two figures are similar there must be a certain correspondence between these elements composing them. Try to think how we can clearly put our idea. To every line in the one there must be a corresponding line in the other; to every point in the one there must be a corresponding point in the other.

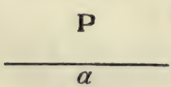


FIG. 62.

Now in mathematics, when we have two figures so related that to every point and every line in one there corresponds

a point and line in the other, then we say there is a *one-to-one correspondence* between these two figures. The definition is often broadened so as to apply to other objects than figures, but we need not here concern ourselves with that, though some of you may some day extend your studies to this wider standpoint. Examine this second figure, Fig. 63, and compare it with the other, Fig. 62, above. Does our definition of one-to-one correspondence apply here? Clearly I can select the line a' as corresponding to the line a, and the point P' as corresponding to P. I can, therefore, get a point and line in the second figure to correspond to every point and line in the first figure. But what line in my first figure corresponds to b'? None. We cannot then find a point and line in our first figure to correspond to every point and line in our second. Therefore, our two figures have not a one-to-one correspondence. Can some one now try to put our idea into clear language?

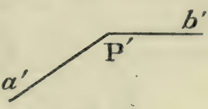


FIG. 63.

ANSWER: When two figures are so related that to every point and line, each to each corresponding, there corresponds a point and line respectively in the other, then these two figures are said to have a 'one-to-one correspondence' with each other.

Let us then agree that the *first* test of two figures being of the same shape or similar is that there be a one-to-one correspondence between them, as defined just now by one of you.

Now you previously said that in order to be similar they must have their corresponding angles equal, and what else?

ANSWER: Corresponding sides proportional.

Let us examine that carefully. Is it necessary to satisfy *both* the conditions just named in order that two figures shall have the same shape? Examine, for example, a rhombus and a square.

FIG. 64.

FIG. 65.

Clearly there is a one-to-one correspondence in points and lines, and there is also exact proportionality of corresponding

lines or sides. But you would obviously not say the two figures are similar. Here then we have hit upon a case which appears to show that proportionality of sides is not sufficient to make the two figures of the same shape. Let us examine briefly another case—a square and an oblong. Here we have corresponding angles equal—the one-to-one correspondence we shall generally find easy to test, so, remembering its necessity, we can for a little omit definite reference to it. It is implied, you will see, in the cases we have to deal with, in the fact that angles and sides correspond. Now the square and oblong have corresponding angles equal, but they, too, are not similar. This is, then, an instance serving apparently to prove that our second test (equality of angles) is not sufficient either by itself. But suppose we always join up the vertices of our figure, and thus convert it into a number of distinct and separate triangles. Now consider for a little this problem

FIG. 66.

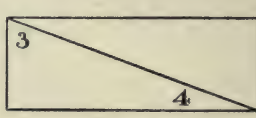


FIG. 67.

carefully. If a figure is built up of triangles, and the angles of the triangles in one are respectively equal to the corresponding angles in the other, are the two figures always and necessarily similar ? Draw some figures and try it, each of you. You know how to make equal angles with your protractors, so that with a little reflection you can easily build up figures, composed of triangles, satisfying the condition of equality of angles.

[A few minutes interval.] Those who have decided hold up their hands. How many think that such figures are always similar ? How many, not ? Let us, for the present, leave the question in this state. Now come back to the oblong (rectangle) and square, and let us apply our test of angle-equality in this new way. As the figures stand, they *have* corresponding angles equal. But have they now ? [Joining a pair of vertices in each.]

On thus dividing up our two figures (Figs. 66 and 67) into triangles, it is clear that corresponding angles are *not*

equal; $\angle 1$ is not equal to $\angle 3$, nor $\angle 2 = \angle 4$. These two figures, then, do *not*, tested in this new way, satisfy our condition that all corresponding angles must be equal.

Now we will, in the first instance, take as our working definition of similar figures—and try to discover later whether our tests are reducible in number to less than three—the satisfying of three tests :—

1. A one-to-one correspondence.

2. The corresponding angles in each are equal, when the figure is built up of triangles. (What this means with figures composed of curved lines we shall discuss at a much later stage.)

3. The corresponding dimensions (sides or limited lines) in each are proportional.

Let us now, for a moment, return to our original figures. Perhaps those who went wrong in doing my little puzzle will by now have discovered the reason.

Suppose the breadth of our oval is two inches; now the new similar figure has to pass through P, about an inch (let us say) above the oval, therefore its breadth will have to be, say, four inches. Consequently, if the figure round our oval is to be similar, by our third test, as we propose to double the breadth we must double the length, so that our new similar figure ought to be—drawn in what way? Thus :—

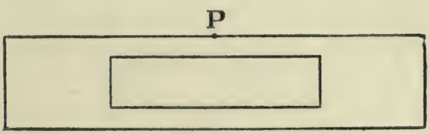


FIG. 68.

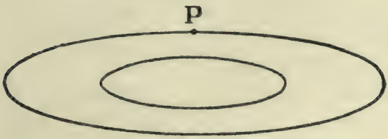


FIG. 69.

Now this experiment serves to show that one's ordinary common-sense idea about similarity and shape is not very exact. It corresponds, indeed, more to what we term in

mathematics 'parallel curves', whose meaning you will gather from your familiarity with the idea underlying the simplest parallel curves, viz. a pair of parallel straight lines whose distance from each other remains constant. The figures you yourselves drew round P were curves *parallel* to the original curves, not *similar* to them: they had not the *same shape.*

It is true that in some cases the figures were drawn correctly; that was owing to the fact, I find, that these particular students had thought about the matter beforehand. I have found, by experiment, that even good mathematicians, if they do not stop to think or reflect, draw, under these circumstances, a parallel curve instead of a similar curve. I gave you these little problems to emphasize the fact that the curves, when they are similar, must satisfy all the three conditions.

The Importance of the Triangle.

Why is the triangle so important a figure in geometry? There are many reasons. See if you can find some of them.

ANSWER: All closed polygons, whether convex or concave, can be decomposed into triangles.

Yes, that is one of the fundamental reasons. Any others?

ANSWER: The sum of the three angles is always the same.

Is the triangle the only figure of which this is true? If not, this is not, by itself, a reason for regarding the triangle as more important than the others.

ANSWER: It is not.

Is what you stated true for four-sided figures?

ANSWER: Not all of them.

Let us examine this point. Do all four-sided figures have this property, that their angles together always come to the same amount?

ANSWER: Yes, they do.

Then that won't really do as a reason to distinguish the triangle from other figures. But it is a good attempt. Perhaps some one else can give me another reason. I want a reason which distinguishes the triangle from all other figures.

ANSWER: If you know the sides, you know all about it.

Is that true of any other polygon? No, it is only true of

the triangle. That is an important reason. We have now these—(1) that every polygon can be divided up into triangles, (2) it is only a polygon of three sides, of which, once the sides are known, we know everything else—thus we should know the angles. So it would appear that the whole of straight-lined (plane) geometry turns ultimately upon investigating the properties of triangles.

. . . .

The last student has brought out another way of stating our fundamental fact. Suppose I ask you to form a figure of three sides. In this figure (showing a triangle of jointed rods), one rod is about 18″, another rod is 12″, and another is 10″. These are the three lengths (separating them). It is putting our fact in another way to say that *if you are given the three rods to build up into a closed figure you can do it in one way only.* If you put the three rods together thus and pin them at the joints thus, you soon begin to feel quite certain that there is no other (closed) figure but this one that can be built out of them. What sort of property will this figure have? Can I alter its shape? No. What do you call such a figure as this whose shape you cannot alter?

ANSWER: A rigid figure.

That is to say, given three rods, put them together so as to form a triangle and they form a rigid figure. Now is that true of four rods or not?

ANSWER: No.

In what way can we alter a figure of four rods? (showing a model of four jointed wooden rods). Take it in your hands and try. . . . Clearly it can be altered in all sorts of ways—thus—and thus. To be given, then, the lengths of four rods is not sufficient to form the figure. In other words, four lines are not sufficient data to form a rigid figure. What other measurements, what other things would be required here to ensure that this should be a *rigid* figure? Think of that for a minute or two.

A bicycle frame.

For example, consider this bicycle frame, *ABCD* (Fig. 70).

The frame of a modern bicycle is a quadrilateral—like this. This in itself is clearly, we have seen, not a rigid

figure. If it were built up thus by means of bars merely pinned together at the joints clearly it would collapse. What is done, then, in order to make the figure rigid? How many supports would be necessary in order to convert this quadrilateral into a rigid figure?

ANSWER: A diagonal.

Then, indeed, it would be a perfectly rigid figure. [The question of deformation out of its own plane is ignored.] But note that by inserting a diagonal you at once convert it into a number of our fundamental rigid figures, namely triangles, *so that the simplest way of making any polygon rigid is to convert it into a number of triangles*, the triangle being a simple, rigid figure, the fundamental figure. (Here a simple application was made to bridge-building.)

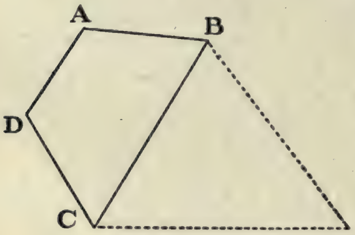


FIG. 70.

As a matter of fact, how is the rigidity obtained in an ordinary bicycle?

ANSWER: By *fixing* the joints (by brazing them together).

How many joints, theoretically, would suffice to fix it?

ANSWER: One.

Not for practical purposes, but one is sufficient theoretically. Then how many measurements fix our quadrilateral completely?

ANSWER: Five.

For instance, four sides and a diagonal, or four sides and any one of these angles. Thus.

Will *any* five elements (sides or angles) do in this figure? I will leave you that problem to think out. . . . If we fix the shape and size, then of course we fix it altogether. *Now will any five measurements fix the shape and size of a quadrilateral?* I mean in this sense, that, given these measurements (e.g. it might be four sides and one diagonal), and of course the order in which they are to be taken, one figure and only one can be constructed from these five measurements. A statement of the order in which the rods are to be fixed I shall call a *qualitative description*. It is quite

different from the lengths themselves, which are *quantitative measurements.*

To get our reply, some investigation is necessary. Try a variety of parts or elements. Try four sides and one angle, three sides and two angles, two sides and three angles, &c.

Gradual building up of the figure.

Suppose first of all we lay down our side AB, then fix the angle ABC, Fig. 71. What else do we need? Let us try one side and four angles, thus making our five measurements, or data, or elements.

Here is AB, then $\angle ABC$, then, say, $\angle BAD$. Remember the lengths of AD and BC are not given. We are given only one side, AB. With these particular five data it soon becomes clear that we cannot fix up our figure. What sort of figure would result if we only knew the remaining two angles? . . . A series of figures like the accompanying.

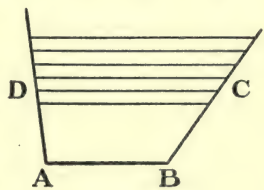


FIG. 71.

Subsequent discussion and attempts to construct the figure brought out the fact that the five data were not *independent*, for the four angles make up 360°, and therefore three angles are equivalent, as data, to four. Hence the five dependent data reduce to four independent data. Similar discussion follows respecting a triangle, and the consequent emergence of *similar* triangles, similar quadrilaterals, &c.

Tests of similarity follow. Attempts to superpose similar figures lead to discussion of *optically* similar figures—figures similar and similarly situated. Then comes the discovery of the simpler Euclidian methods of proof in the beginning of Book VI, dealing with similar figures based on the discussion of conditions of similarity of *triangles.* Another reason emerges for the importance of the triangle, viz. as regards the two tests of similarity (proportionality of sides and equality of angles) in the case of the triangle, each implies the other, i.e. the two tests reduce to *one* in triangles.

Then follow the relations between Euclid I. 4, 8, and 26 and the corresponding theorems in Book VI. Supply one linear dimension in each case and these Book VI theorems

reduce to the Book I theorems, in which case the *similar* figures become *identical* figures.

Typical problem for discussion and homework :—

In a triangle two sides are 2″ and 3″ respectively, and the angle opposite the 2″ side is 30°. Construct as many triangles as you can, fulfilling the conditions, and examine the problem by Algebra and Book II of Euclid, or by Trigonometry.

NOTE.—In treating of lines and other magnitudes in 'proportion', it should be added that, for elementary geometry, incommensurable magnitudes are assumed to be replaced by commensurable magnitudes approaching to the others as closely as we please (by the usual arithmetical methods).

In the Euclidian or other formal treatment of geometry [1] the valuable notion of similar figures is delayed so long (until the sixth book is read !) as to be actually never reached at all *as a definite and useful conception* by a great number of our scholars : simply because they complete their geometrical studies with the first few books. Yet the idea appears to be present in a vague form at a very early age indeed as exhibited in the early recognition of external objects in *pictures*, and only needs the attention to be concentrated upon it to emerge into full clearness as a defini-

[1] Clifford, in his treatment of geometry in *The Common Sense of the Exact Sciences*, starts with the axiom 'that it is possible to have things of the same shape but of different sizes (see p. 47) and subsequently defines thus —'Two bodies are of the same shape, if to every angle that can be drawn on one of them there corresponds an exactly equal angle on the other' (p. 66). One would like to see this book more in the hands of teachers than it appears to be : so eminently suggestive is the spirit in which it is conceived. Perhaps I may be pardoned for adding that Clifford's treatment of the subject appears to neglect the part played by the constructive imagination (in its conceptualizing aspect) in the formation of precise geometrical ideas (see footnote on p. 48 by the Editor, Karl Pearson, with whose criticism I am in general accord). The truth of the matter has been well put by Caird, who, speaking of Mathematics says, 'Its own special objects are produced by the very act of mind that defines them : i. e. we give rise to those objects by putting together conceptions in an arbitrarily determined way. . . . Thus, 'the circle is not given as an object *before* the definition of it, but comes into existence in virtue of it' (CAIRD, *The Critical Philosophy of Kant*, Vol. I, p. 136). It is to be observed that this view does not conflict with a simultaneous opinion that the construction of geometrical science would have been impossible without sensible experience of geometrically extended objects.

tion and thence become the basis for a sound knowledge of elementary surveying, &c.

Long experience has convinced me that the Euclidian or any formal method of treating proportionals is much too complex and subtle for the intellectual powers of the average schoolboy and girl. The very notion of incommensurability of magnitudes forms itself a difficulty of no mean order, but when to this is added a method of comparing ratios refined and comprehensive enough to include philosophically incommensurable and commensurable magnitudes alike, the difficulty rises into an impossibility for the majority—and the great majority. The old results of lack of understanding inevitably follow—dogmatism in the teacher, blind mechanism in the pupils. Moreover, it is to be noted that, beautiful and ingenious as the method is, we, by universal omission of the *Fifth Book*, whereon the Sixth Book is founded, find ourselves, after all, compelled to rely on algebraic theorems for the various truths in proportion constantly employed in Book VI. The demands of consistency ought surely long ago to have determined the entire replacement of this treatment of proportion by the modern arithmetical mode of appeal to a unit of measurement, which is perfect for commensurable magnitudes, and for others can be applied with a degree of approximation unlimitedly close to the truth. Here the demand for a philosophically perfect theory and absolute precision is surely quite out of place. The custom retained in many schools —though happily now discarded in most—of still adhering to Euclid here would appear to be nothing but a pure survival of mediaevalism in mathematics, to wit, the practice almost universal amongst mathematicians—though not, of course, among business men—up to a very late date (witness Newton's *Principia*, 1687), of stating results in the form of a proportion amongst four terms, where the moderns would simply use a single equation, based on the conception of a unit. Wallis, in his Algebra (1686) is said to have been the first to employ systematically a unit of measurement in order to express proportions by a single formula. The endless wrangles that arose in mediaeval times, among eminent philosophers and mathematicians, concerning the Euclidian treatment of proportion should have alone sufficed to warn us of the utter unsuitableness of the idea for school children

(such discussions may be found, as late as 1665, in Barrow's *Lectiones Mathematicae*). The idea, on which the comparison (the ratio) of two magnitudes is founded by the Greek mathematicians (the formation of an infinite scale of multiples of each, and observation of the mutual relation of these), is, after all, but a method of approximation capable of being carried out, in conception, to an indefinite extent; and this property of indefinitely close approximation the modern unit method also possesses, while being incomparably easier to apply for ordinary purposes.[1] The method is a refinement of the idea underlying the common mode of comparing two magnitudes, as when we say, 'This object is *between two and three times* as big as the other,' a notion which, we admit, appears to be much more fundamental than the conception of a unit capable of simultaneously measuring both objects under comparison. *For practical purposes,* we might say that the two methods differ in spirit and applicability as much as would the comparison of the heights of a father and his child by taking the child as a unit to measure the father, differ from their comparison by simple application of a footrule to each. One method requires new standards of measurement for every two objects compared: the other fixes upon one definite unit with which to measure all.

The use in many schools of the dot-method in Rule of Three is a similar historical survival whose original justification has vanished.

[1] I say 'for ordinary purposes': seeing that the modern powerful 'General Conception of a Number' is based directly upon, and is a further refinement of, the Euclidian treatment of incommensurables (see Harnack's *Differential and Integral Calculus*, English Translation, Williams and Norgate, 1891, page 9, § 6). My purpose in the text is simply to point out that its very refinement and power, philosophically, make it unfit for school treatment.

CHAPTER XIV

BRIEF NOTES OF LESSONS

GEOMETRY (DEMONSTRATIVE AND PRACTICAL), INDICES, EQUATIONS, CALCULUS OF DIFFERENTIALS, ETC.

1. *Geometry (Practical).*

PROBLEM, determination of a circle :—

The teacher had shown clearly, with plenty of questioning, that three points determine a circle. The basis of the proof lies in two applications of the proposition that the locus of points equidistant from two given points is a straight line. The problem now was : *given a part of a circle, find the rest of it.* An actual circular arc was drawn on the board, and the pupils invited to try to complete it, *with the smallest number of operations.*

First attempt of a pupil :—

In the figure the two extreme points A, B, of the arc were taken, the pupil then found, by application of the compasses, the middle, C, of the arc ; by further application of the compasses and use of the straight edge he then bisected BC and CA at right angles, and obtained the centre. Number of times compasses used, plus straight edge used, was fifteen. This pupil, in bisecting the lines at right angles, did not notice that two openings of the compass were sufficient : he repeated the same construction above and below, thus using the compasses four times instead of twice.

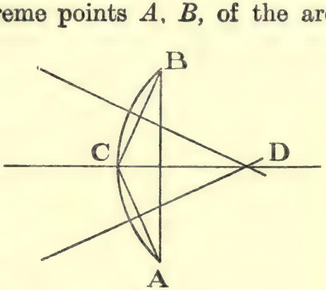


FIG. 72.

Second attempt of pupils :—

Another pupil took any three points on the curve and very much simplified the number of operations; result, seven operations.

Finally, another pupil pointed out that it was unnecessary to join the chords, and that three uses of the compass sufficed in general. This reduced the number of operations to five. Hence the proposition :—*given a part of a circle, the centre can be found with five operations with ruler and compasses, and the circle can be completed with six.* (See also p. 180.)

Problems for class— :

Definite arc given, to complete the circle and find the radius.

Any three points taken, circle to be completed and radius to be found in inches.

Two parallel chords of a circle given, but not the circle; find the centre.

Given any two parallel lines, can a circle be put round their four extremities, and, generally speaking, can a circle be put round any four points; if not, why not? If a circle can be put round four points, what special relation must there be between those four points?

It is highly important that pupils should both grasp the proof and actually go through the operations required, and that they should feel that the essence of practical geometrical drawing lies in intelligently economizing the operations necessary, consistently with the attainment of accuracy (viz. the aim of mathematics is ultimately a minimal problem).

2. *The relation between Geometrical Drawing and Geometrical Science.*

The former is an art, the latter is a science. The former is practice, the latter is theory. Each must have its due: each a systematic share in the training. But the rational alliance between them must be brought into prominence. In subjects so intimately connected as these two this can hardly be done, with any reasonable degree of success, unless one teacher takes both subjects. In the early stages of geometrical teaching the two will develop in apparently indissoluble connexion. But gradually each aspect should gain independent strength and finally they should emerge

as separate branches of study with distinct objects, and yet constantly feeding one another and never for long losing touch. Ultimately, the theory must be readily applicable to practice without ceasing to be scientific, and the drawing must be rationalized without ceasing to be thoroughly practical. To create and preserve independence, and simultaneously to foster the alliance is a real difficulty which only a capable and experienced teacher of mathematics, versed in both aspects, can be expected to overcome. It too often happens that the mathematical teacher is expert in the scientific aspect only. He ought also to have the essential qualities of a reasonably good draughtsman, in the use of his instruments. The training colleges and universities should aim at the production of teachers of this type. Otherwise, a veritable chaos will obtain in mathematical teaching; and, in the minds of pupils, a confusion of thought springing inevitably from lack of system and resulting in obscurity and haziness, will for a time be the unexpected and pernicious fruit of the present reform-movement.

The object of geometry, as science, is to explore the properties of space under the guidance of systematic logical machinery gradually perfected by mankind for this purpose. Without this logical paraphernalia of definitions, axioms, postulates, &c., geometry would for ever remain an empirical art. As an educational discipline, its distinctive value would then be largely if not entirely lost. To geometry as pure science it is a matter of indifference whether its figures can or cannot be actually and accurately constructed with the instruments whose use is postulated. Their use for science is satisfied fully if they exist *in posse*, they need not exist *in esse*. The driving impulse in the elaboration of pure geometrical science is sheer aesthetic wonder at the complexity of spacial properties: it is an ideal world, created, so far as the machinery of investigation is concerned, by the geometrician himself. The geometrician, pure and simple, is ultimately an artist, but to become an artist he must serve his apprenticeship to science. The importance of practical applications of geometrical science as reacting upon the development of the science itself has been sufficiently emphasized. The other aspect of the truth must not, however, be minimized, still less forgotten—that science must periodically pass through phases in which the question

of practical applicability is absolutely and utterly ignored, when it is cultivated from motives of disinterested and, shall we say, divine curiosity, wonder, and admiration, when we might almost declare that the searching mind struggles in vain to rise beyond the limitations inherent to space as we experience it here and now—struggles in vain, yet in the very effort partly creates, partly discovers truths of sublime beauty and generality which would never have revealed themselves to a less disinterested spirit. This aspect of geometry as science, as it is true of the race, so must it have its counterpart in the school: otherwise we are false to our principle of parallel racial and personal development, and, what is of still greater import, unmindful of the lessons of experience in the history of mathematical teaching.

This alternate absorption of science in the arts of practical life and entire freedom from them appears to be a law of development of all sciences and arts. Sometimes the period of alliance or divorce has lasted for generations, and even centuries, in certain nations, leading ultimately to either gross and materialistic empiricism or to fertile paths of research: but so tenacious is humanity in the long run of achievements once made that though the nation has suffered the race has gained. The Chinese mathematics may be instanced as a type of the former extreme, and the Byzantine as a type of the latter. Or it may happen that—as in modern times—the two aspects (theory and practice: science and art) simultaneously progress, acting and reacting upon each other, but the two lines of progression may largely develop in different minds—herein an idealist, ignorant and perhaps contemptuous of practice, therein a realist, equally ignorant and contemptuous of theory. But here again, though the individual suffers by one-sidedness and excessive specialism, the race gains.

Bacon tells us that we can only conquer nature by obeying her laws. It should, therefore, be possible, once the law of development is recognized—the alternate action and reaction of science and art—to mould our education accordingly, and, while attaining substantial progress in each direction, avoid, by conscious and combined effort, the sacrifice either of nation or of individual.

The application to school education is deemed obvious after some reflection without further labouring the point.

Bearing in mind this balanced attitude, teachers, whose excess is in the direction of theory, may derive profit by reflection upon the following remarks of Steiner, the great Swiss geometrician (1796–1863). ['Die geometrischen Constructionen, ausgeführt mittelst der geraden Linie und eines festen Kreises, als Lehrgegenstand auf höheren Unterrichts-Anstalten und zur praktischen Benutzung; von Jacob Steiner, 1833.' Leipzig: Engelmann, 1895. (Ostwald's *Klassiker der exakten Wissenschaften*, pages 64–6.)] The following is a free translation:—

'It appears that in general, up to now, too little attention is paid to the question of geometrical constructions. The customary method, handed down to us from the ancients, is to regard a problem as solved directly it is shown by what means it can be reduced to others previously disposed of. This prevents us from properly judging what a full solution really implies.' [e.g. Let the reader count the number of operations required, with ruler and compasses, to carry out, *ab initio*, such a simple practical problem as the drawing of a perpendicular to a straight line from a point external to it, and let him finally carry out the constructions according to Euclidian procedure. He will then be in a position to appreciate Steiner's attitude.] 'Thus it happens that in this manner constructions are frequently given which, if one were really compelled to rely upon them, and attempted to carry out really and exactly all they imply, would quickly be given up as useless, for one must very soon be convinced that it is one thing to make the constructions by hand with the actual instruments, and quite another—to allow myself the expression—to carry them out with the tongue. It is easy enough to say, "Do this—then this—then that", but the difficulty, one may even say in some cases the impossibility, of carrying out fully highly complicated constructions demands in any proposed problem a searching inquiry into the possible courses of procedure. One must discover which are simplest, or which are most appropriate in given circumstances, and how many of those operations, which the tongue so lightly deals with, can be circumvented, remembering meantime the necessity of avoiding superfluous trouble, or of attaining the greatest accuracy, or of economizing as much as possible the paper on which the construction is made. In a word, the question is: By

what method any geometrical problem, theoretical or practical, can be solved most simply, most surely, or most accurately, (1) in general, (2) with limited instruments; and, finally, (3) under substantial difficulties, what procedure is most appropriate?' Steiner goes on to add that he has discovered scientific methods of solving problems merely with the use of a straight edge and one given fixed circle (i.e. with one use of the compasses only) which are as simple and convenient as by the use of any combination of instruments whatever. Here again we see the mutual support that science and practice may give each other.

3. *Indices.*

In dealing with indices, and in fact as a general educational principle, it is often advantageous to throw the pupils on their own resources by giving them a somewhat more complex problem than customary in order to make them *think and struggle.* Thus, shortly after the introduction of indices—in response to a desire for contractions, for brevity of symbolism—discuss the difference between 3^{2^4} and $(3^2)^4$. Also discuss and contrast, without of course actually calculating the final value, say $3^{5^{2^{4^{3^2}}}}$ and $((((3^5)^2)^4)^3)^2$.

In dealing with the former, ask the pupils to close their eyes while it is written on the blackboard and cover the figures up. Then, with opened eyes, allow the figures *to be gradually uncovered from right to left.* Proceed similarly with the second expression. Emphasize the value of symbols and the extraordinary saving of time effected by the discovery and use of the index notation.

4. *Simultaneous Equations.*

In solution of simultaneous simple equations, whether curve-tracing is simultaneously given or not, in order to keep clearly before the pupils' minds the fact that their solutions must satisfy both equations simultaneously, the pupils should be made to combine with a bracket the two equations throughout, in the early teaching of this subject, until the essence of the idea is sufficiently familiar to them. The same, to a less degree, should be done with simultaneous Quadratic Equations. For example :—

$$\text{I.}\ \left\{\begin{aligned} x+2y&=5\\ 3x-\ y&=1 \end{aligned}\right\}$$

$$\therefore$$

$$\text{II.}\ \left\{\begin{aligned} x+2y&=5\\ 6x-2y&=2 \end{aligned}\right\}$$

$$\therefore$$

$$\text{III.}\ \left\{\begin{aligned} x+2y&=5\\ 7x\qquad&=7 \end{aligned}\right\}$$

$$\therefore$$

$$\text{IV.}\ \left\{\begin{aligned} x+2y&=5\\ x\qquad&=1 \end{aligned}\right\}$$

$$\therefore$$

$$\text{V.}\ \left\{\begin{aligned} 1+2y&=5\\ x\qquad&=1 \end{aligned}\right\}$$

$$\therefore$$

$$\text{VI.}\ \left\{\begin{aligned} 2y\qquad&=4\\ x\qquad&=1 \end{aligned}\right\}$$

$$\therefore$$

$$\text{VII.}\ \left\{\begin{aligned} y\qquad&=2\\ x\qquad&=1 \end{aligned}\right\}$$

In each step it should be made clear that each of these pairs of equations is logically equivalent to the preceding pair, and that the whole process is a reduction of a complicated pair to $y=2$ and $x=1$, and that, except on the score of simplicity, there is really no difference between the first and the last. Each represents a mode of fixing a point by the intersection of a pair of straight lines, the position of the straight lines simply differing in simplicity. In the case of simultaneous quadratics, emphasize the same principle. Here we reduce a combination of conics and straight lines, or two conics, to two pairs of simply situated straight lines. It is recommended that sometimes the corresponding pairs of straight lines, viz. the straight lines represented by I, II, III, IV, V, VI, and VII, &c., should be actually drawn throughout. Similarly in the case of solution of simultaneous quadratics, reconsideration should be given to these fundamental principles, if and when this type of equation is reached.

5. *Note on a lesson in the Differential Calculus.*

The lesson was on the application of early notions and elementary rules of differentials to calculation of small errors.

The pupil already knows the rule : $dy = 2x\,dx$ if $y = x^2$.

Consider the function $y=x^2$ (illustrated by a square, of side x ft.).

1. Give a number of simple numerical problems, worked out by ordinary arithmetic (see Fig. 73).

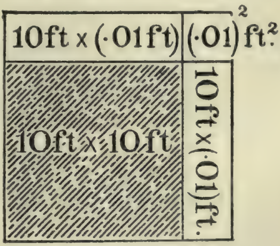


Fig. 73.

2. Gradually generalize these examples (see Figs. 74 and 75).

3. Until, finally, the pupil is able to discuss and state the rule for himself in the case where the dependent variable is the square of the independent variable; for example, calculating small errors made in the area of squares due to small errors made in the measurement of the side, namely, the rule :—

(*a*) Express the magnitude whose error has to be calculated as a function of the independent variable whose error is known.

(*b*) Then by differentiating obtain the differential of the dependent as a multiple of the differential of the independent variable.

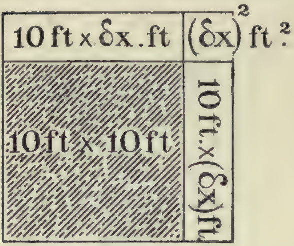


FIG. 74.

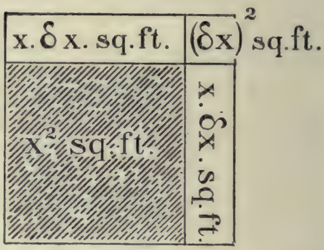


FIG. 75.

NOTE.—The two ideas, (*a*) and (*b*), involved in the rule, should be clearly distinguished and emphatically impressed upon the pupils as a conscious method of dealing with this and other similar problems.

Throughout, although the process is lengthy, the students should calculate both by a general figure and by analysis not only the *approximate* error in the dependent variable, but also the error made in regarding *this* as the approximate error. (Assuming one knows the exact error in the dependent variable.) For it is as important to know the degree of accuracy of the approximate error as to know that error itself. Not otherwise can one be sure that the result is correct to so many decimal places. This is important to bear in mind as a general principle in the calculation of small errors.

In fine, impress the mind repeatedly with the idea of small quantities of different orders, 1st, 2nd, . . ., in this connexion also, i.e. as gradually decreasing errors, here.

4. After these should come a number of stiffer numerical and general examples.

5. Then the pupils should be in a position to discover easily for themselves the rule for the case of $y = x^3$, and illustrate geometrically.

6. Then rise to the n^{th} power, and after this there should be no difficulty in extending the idea to any simple function whatsoever within the pupils' knowledge; for example, trigonometrical ratios, &c.

General Note.—In the above process it will be seen that the method originally used for obtaining the actual differential of x^2 with respect to x—a knowledge of which was assumed at the start—is repeated in this problem. This is not a waste of time, but an important principle to apply continually in mathematics, repeating proofs even in detail, when new applications have to be made—for example, it should be done in connexion with the application of differential co-efficients or differentials to velocity, small errors, tangents, &c.

Finally, the above process with tolerable certainty assures a pretty thorough understanding of the principle by the pupil because he has gradually been led to discover it for himself, and the subsequent application impresses it upon him so that he cannot readily forget it.

6. *Heuristic rationalization of the 'common-ratio' artifice in Proportion, and simultaneous illustration of the fundamental logical principles of simultaneous equations.*

No better instance perhaps can be given of the indecent haste to attain results that will pay in examinations without any real attention to the conduct of the understanding than the mode in which textbooks and teachers generally present to the pupil the following artifice in algebraic manipulation. I refer to the usual method employed in such a problem as this :—

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove each of these ratios equal to $\left(\frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}\right)^{1/n}$, where p, q, r, n are any quantities whatsoever.

I take the following solution from a textbook in wide use and of high reputation :—

'For let $k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$; then

$$kb = a, \ kd = c, \ kf = e;$$

therefore $p(kb)^n + q(kd)^n + r(kf)^n = pa^n + qc^n + re^n$;

$$\therefore \ k^n = \frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}, \quad \therefore \ k = \left(\frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}\right)^{1/n}.$$

And this artifice (amidst such a complication of symbols, too, as would certainly prevent the ordinary student from grasping its spirit) is suddenly introduced without a single word of explanation—or indeed any hint that such is desirable—as to the reasons *why* the method is so successful, wherein consists its real power and what are the limitations to its use. Very beautiful and very useful the artifice certainly is—but only really so to those who grasp its spirit. To the average schoolboy it appears as a kind of miracle, and, after being shown a few examples of it, he himself, in purely mechanical fashion, successfully solves by its aid a large number of problems carefully constructed beforehand in such wise that precisely the same method applies. What is the educational value of this ? Let us wait a moment. Give him the following problem, requiring but a slight modification of the same principle :—

$$\text{If } \frac{a}{b} = \frac{b}{c} = \frac{c}{d}, \text{ prove } \frac{a-d}{b-c} = \frac{bd(b+c+d)}{c^3}.$$

He fails hopelessly—unless an example precisely similar to this, again, has been mechanically worked for him and mechanically accepted. *Here, as so often, teachers and textbooks solve for the pupil a difficulty before he has recognized it as such.* Now, properly presented, this method supplies a very valuable discipline indeed to the reasoning powers, and can be made to spring naturally from first principles, instead of assuming the guise of a mere artifice. Thus, in the above and like problems, the attention should be directed to their elucidation by help of an appeal to the primary logical principles underlying the solution of equations.[1]

[1] A careful discussion of which was, I believe, first introduced into our School treatises on Algebra by Professor Chrystal. He justly complains

As the principles involved are of great importance, the method is worthy of serious and careful discussion and study. The following brief notes of lessons, if carefully studied, will perhaps be found useful in essentials of treatment.

(i) Begin with revision of the geometrical interpretation of, say, $x+y=2$ (a straight line), and the ideas (variable, constant, &c.) involved.

(ii) Suppose that all we know about four quantities a, b, c, d is that $\frac{a}{b}=\frac{c}{d}$. How many of these four variables may take any value we please; i.e. how many are 'independent' variables? Let the pupils try.

Thus let $a=5$, $b=3$, $c=-1$. At this point the pupils see at once that $d=\frac{bc}{a}=-\frac{3}{5}$, and so is thereby determined.

Clearly any three may be independent, but the fourth is dependent.

A table may now be formed of the discussion (as a model for future similar discussions) :—

Number of variables I	Number of *independent* equations II	Number of *dependent* variables (i.e. variables whose value can be calculated when those of the rest are assigned) III	Number of *independent* variables IV
4 (a, b, c, d)	1 $\left(\frac{a}{b}=\frac{c}{d}\right)$	1 $a=bc/d$ or $b=da/c$ or $c=ad/b$ or $d=cb/a$	$4-1=3$ viz. a, b, c or b, c, d or a, b, d or &c.

(iii) Again, gradually increasing the number of variables: consider $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$.

that 'there are few parts of Algebra that are treated in more slovenly fashion in elementary teaching' than the logic of the derivation of equations.

Here the pupils should be able, with guidance, to develop a similar set of results.

How many independent equations or facts are deducible from $A = B = C$? Only two, which we may take as $A = B$, $B = C$; $\therefore A = C$; *or* we may take $A = C, B = C$; $\therefore A = B$; and so on. Generalize this result.

Thus our table (using I, II, III, IV to denote the columns as above enumerated), after some numerical testing, will run :—

I	II	III	IV
6	2	2	$6-2=4$.

By trial it is found that the two dependent variables may not be *any* two, e.g. a and b cannot both be taken as dependent, for $\frac{a}{b} = \frac{3}{4} = \frac{6}{8}$ does not enable us to calculate both a and b. Nor can c and d; nor e and f. The rule is obvious and the reason for it. Then a discussion may follow as to the possible combinations, thus :—

I	II	III	IV
a, b, c, d, e, f	$\left(\begin{array}{l}\frac{a}{b} = \frac{c}{d}\\ \frac{c}{d} = \frac{e}{f}\end{array}\right)$	$a, c\left(\begin{array}{l}a = b\,.\,e/f\\ c = d\,.\,e/f\end{array}\right)$	b, d, e, f
		or	
		$a, e\left(\begin{array}{l}a = b\,.\,c/d\\ e = f\,.\,c/d\end{array}\right)$	b, c, f, d
		or	
		&c.	&c.
		[Any 6, 2 at a time, except ab, cd, ef, i.e. $\frac{6\,.\,5}{1\,.\,2} - 3 = 12$ sets.]	

Give also plenty of concrete examples (measurement).

(iv) Consider $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$.

Here :—

I	II	III	IV
8	3	3	$8-3=5$,

and so on until the logical principles involved become clear.

Applications:—(*a*) If $\frac{a}{b} = c$, prove $\frac{a+c}{b+1} = \frac{a-c}{b-1}$ .... (1)

Apply the principles above elucidated. *Two* quantities are independent. (1) is a *conditional* equation as it stands. If we replace the dependent variable, say, a, in terms of b and c in the conditional equation (1), what difference will it make to the nature of this relation? Try it.

$$a = bc, \text{ and } \frac{a+c}{b+1} \text{ becomes } \frac{bc+c}{b+1} \text{ or } c.$$

Also $\frac{a-c}{b-1}$ becomes $\frac{bc-c}{b-1}$ or also c.

The conditional equation becomes an *identity*. This should be clear also otherwise. Because if we replace all variables in terms of *independent* variables, say b and c, then, as b and c being independent may assume any values we please, the two sides of the relation (1) are now to be true for *any* values of the letters or variables they contain. They must, therefore, be identities.

(β) Again: If $\frac{a}{b} = \frac{c}{d}$, prove $\frac{ac}{bd} = \frac{a^2}{b^2}$.

As before, the Table is to be filled up, thus:—

I	II	III	IV
4	1	1	3

Choose for the one dependent variable, say, a.

Then $a = \frac{bc}{d}$. Our conditional relation becomes

$$\frac{\frac{bc}{d} \cdot c}{bd} \text{ to be proved } = \frac{\left(\frac{bc}{d}\right)^2}{b^2},$$

$$\text{i.e. } \frac{c^2}{d^2} \text{ to be proved } = \frac{c^2}{d^2}.$$

(γ) Again: If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$,

prove $$\frac{a+2c+3e+4g}{b+2d+3f+4h} = \sqrt[4]{\frac{aceg}{bdfh}}.$$

The table of variables and equations here is :—

I	II	III	IV
8	3	3	$8-3=5$.

Choosing a, c, e for dependent variables, we have :

$$a = b(g/h),$$
$$c = d(g/h),$$
$$e = f(g/h),$$

and the two sides of our *conditional* relation which is to be verified assume respectively the form :

$$\frac{b\left(\frac{g}{h}\right) + 2d\left(\frac{g}{h}\right) + 3f\left(\frac{g}{h}\right) + 4g}{b + 2d + 3f + 4h}$$

and

$$\sqrt[4]{\frac{b\left(\frac{g}{h}\right) \cdot d\left(\frac{g}{h}\right) \cdot f\left(\frac{g}{h}\right) g}{bdfh}},$$

which, of course, must be identically equal. They are, in fact, both reducible to g/h.

(δ) Can any pupil state in clear language the method we have developed ?

(ϵ) Can any pupil suggest a more rapid method, by utilizing the principle (so often found useful both in algebra and geometry) of *symmetry* ?

In the above examples and others done on the same models as exercises the pupils will, some of them, not fail to have noted that certain combinations have repeatedly to be written down (e.g. $\frac{g}{h}$ in the last example). It has, let us assume, with good teaching been the custom to utilize algebraic symbols to shorten labour. Thus the pupils should naturally here suggest the saving of considerable trouble by the simple device, so often already used, of *substituting some single letter for the combination repeatedly used.* This once done, the rest follows with little difficulty. It will, by careful observation, be gradually seen that many, if not most of the most complicated algorithms and processes in mathematical analysis are, as Chrystal has well said, merely 'the creation of a commonplace'. But, though this is so,

it must be added that the mere introduction of an additional symbol into analysis sub-consciously implies the simultaneous introduction of powerful processes already in existence and created by the labours of previous mathematical workers. Thus even an Euler could justly confess that he often felt his pen was, in some mysterious way, more powerful and pregnant than himself during his mathematical investigations.

Thus suppose here we replace, *to save trouble,* g/h by any single letter, say ρ.

Then, inevitably, we have

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} = \rho.$$

How, it will naturally be asked, does this affect our table ? The position seems worse, as we have another variable to deal with, and yet, there must be economy somewhere, as we saw that the introduction of this ρ saved us trouble in the writing of so many copies of the complex symbol g/h.

Our table stands thus :—

I (variables)	II (independent equations)	III (dependent variables)	IV (independent variables)
9	4	4	$9-4=5$

Note that the number in the fourth column will always remain unaltered by the introduction into these problems of a new letter or variable. Why ?

The question now is, *Which quantities shall we select as our independent variables ?* A little discussion soon shows that, if common-sense is used, there is not much difficulty in selecting the best combination. Thus we specifically inserted ρ as useful, therefore ρ must be one of our independent variables. Clearly it is then very easy to replace the numerators a, c, e, g in terms of the denominators b, d, f, h, with the help of ρ.

Note also the beautiful symmetry of this choice.

Our procedure is then easy.

$$\begin{aligned} a &= b\,.\,\rho \\ c &= d\,.\,\rho \\ e &= f\,.\,\rho \\ g &= h\,.\,\rho \end{aligned}$$

Substituting, the conditional equation becomes :—

$$\frac{b\rho+2d\rho+3f\rho+4h\rho}{b+2d+3f+4h} = \sqrt[4]{\frac{bdfh\rho^4}{bdfh}}$$

i.e. an identity.

(ζ) *How to state the Principle of the Method.* An excellent exercise for the pupils is to try to state clearly the principle of the method, viz., the Principle of the Minimum Number of Variables. It may be stated thus :—Suppose we have given a number of independent relations between certain variables, and that some other relations, conditional thereupon, are to be established between those variables. By means of the given relations, express all the variables involved in terms of the *smallest possible number of variables*. Make the appropriate substitutions required in the conditional relations, and they will, if valid, reduce to identical equations (or identities).

(η) Further examples on the 'ρ' method.

(θ) Further tests of comprehension of the principle (ζ).

If $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d}$, prove $\dfrac{ab+cd}{ab-cd} = \dfrac{b^2+d^2}{b^2-d^2}$.

Many pupils will begin thus :

$$\text{Let } \frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \rho \quad \therefore \quad \begin{cases} a = b\rho, \\ b = c\rho, \\ c = d\rho, \end{cases}$$

and substitute these values for a, b, c. The result is unexpected : no identity results. What is the reason ? Return to the fundamental principle by forming a Table.

	I	II	III	IV
Original equations	4	2	2	2
Equations with ρ	5	3	3	2

It is then possible to put all variables in terms of *two* only. Upon inspection of the unexpected result reached

above it will be seen that this condition of the 'minimum' number of variables has not been fully carried out.

Continue then :

$$\begin{cases} c = d\rho, \\ b = c\rho = d\rho^2, \\ a = b\rho = d\rho^3. \end{cases}$$

Here we have all variables in terms of *two* (d, ρ), which is the minimum required.

Our conditional relation now really reduces to an identity on making these substitutions ; and so on.

(ι) Finally, what is the principle involved here ?

'If any number of magnitudes are in continued proportion, they may all be expressed in terms of any one and the common ratio.'

Further Examples.

(κ) Subsequently connect this with the idea of a Geometrical Progression or Series :

$$a + a\rho + a\rho^2 + a\rho^3 + \ldots$$

and Simpson's Rule for Quadrature of Curves : also, of course, with Variation.

7. *Note on approximate numerical solution of equations.*

Here a hint from the history of mathematics is again useful. Vièta's method of approximation, simple in its essence, may with much advantage be introduced into the treatment of Quadratic Equations. It has the advantage, also, of early familiarizing the mind with the valuable idea of small quantities of different orders, and thus also of laying sure foundations for subsequent study of the Calculus.

Only a brief sketch is here given.

It is assumed that by this point some simple notions of curve-tracing have been given. A few examples, with an appeal to intuition in curve-tracing or otherwise (the method is not dependent on curve-tracing and may be introduced independently), will evidence the truth of the theorem that 'In general, if an expression in x is $+ve$ for one value of x and $-ve$ for another, then it must be zero for some value of x between these two.'

e.g. To discover a $+ve$ root of $x^2 - 2x - 2 = 0$ try $x = 1$, $x = 2$, $x = 3$, &c.

$x = 3$ gives for $x^2 - 2x - 2$ the value 1.
$x = 2$,, ,, -2.

$\therefore$ we may assume a root between 2 and 3, and nearer 3.

Let this root $= 3 + s$ where s is small.

$$\therefore\ 9 + 6s + s^2 - 2(3 + s) - 2 = 0.$$

$\therefore$ *neglecting* s^2 *in comparison with* s, $s = -0{\cdot}3$ (nearly).
$\therefore$ a second approximation is $x = 2{\cdot}7$.

For a third approximation let $x = 2{\cdot}7 + s$.

Then, proceeding as before, we find $s = 0{\cdot}03 \ldots$ $\therefore$ the third approximation is $x = 2{\cdot}73$. And so on, to any degree of accuracy required.

The value of the process is that it is *self-adjusting* or self-correcting, and the verifiable result justifies any unproved assumptions that may have been made. This is a most valuable characteristic.

Query : Find an easy criterion for the magnitude of the error in the root.

The method may be applied to numerical algebraical equations of any degree—another valuable feature—though tedious in general.

The teacher will also notice that the method is easily applied to the extraction of square, cube, and higher roots, corresponding to the equations $x^2 = 1$, $x^3 = 1$, &c.

Further refinements and additions to the principle (Vièta's, see any history of mathematics) have been made—by Newton, Fourier, Horner, &c.—but, except for pupils specializing on mathematics, the above is sufficient, beautiful and powerful though Horner's process is.

It is assumed that the whole subject, thus briefly epitomized, would be presented in the heuristic way recommended throughout.

CHAPTER XV

ORIGIN AND DEVELOPMENT OF MATHEMATICS

1. *The Intimate Dependence of life-experience on Mathematical Thought.*

THERE are four fundamental characteristics of nature which make our life-experience more intimately dependent than we usually conceive on the creation of mathematical thought. For every material object has *shape* and *size*; every material object is capable of *motion*; and, finally, everything, material or non-material, can be *numbered.*

Broadly speaking, we may say that the existence of mathematical knowledge, at some point or other of its immense range between simple counting and the familiar use of the language of form at the one extreme and the highest reaches of the infinitesimal analysis at the other, is *one* of the conditions—and that not the least important—that has made possible to mankind the development of every kind of science and art, of industry and commerce.

2. *Mathematical Experience: its Origin and Development.*

The influence of mathematics upon other branches of human activity has been briefly alluded to. But this is only one side of the whole truth. If mathematics has aided in developing science, art, industry, commerce, &c., as assuredly have these human activities themselves in turn aided in developing mathematics. No action is merely one-sided. Indeed, mathematical thought has arisen and grown in response to definite impulses, vividly-felt needs of our ancestors. I think we may justly classify these impulses or stimuli into three main groups.

a. The Practical impulse or factor.

First and foremost there is the impulse, the stimulus, one may perhaps say the *necessity* of dominating the natural difficulties constantly presented by the physical world around us. Let us call this the *practical impulse* or factor.

It was in his practical struggle with his physical environ-

ments that our primitive ancestor evolved those simplest and fundamental ideas about number, quantity, and space, once so difficult to grasp, and now so easy. The beginnings of arithmetic and geometry we owe to primitive mankind and early civilizations.

The shepherd who cuts notches on his stick to count his sheep: the traders who evolve (as in China) a system of finger symbolism capable of rapidly expressing any number up to ten thousand[1]: the builder who uses a stick the length of his foot or of his stride as a fairly constant unit of measure (the primitive *foot-rule*): the wheelwright (as in ancient Egypt) who strengthens his wheel with six equal struts, thus perhaps discovering the fact that the radius of a circle goes six times, as a chord, round the circumference: the surveyor, the geographer, and the artist who discover the properties of similar figures in their attempts to picture the earth and the objects upon it—these and numberless other instances make us see to what sources our fundamental mathematical knowledge is due. Moreover, this factor—the practical factor—in the development of mathematics never ceases. In quite modern times mathematics has owed many a striking discovery to the practical needs of the engineer.

FIG. 76.

'Observing that all the operations connected with the construction of plans of fortifications were conducted by long arithmetical processes,' Gaspard Monge (the French mathematician, son of a pedlar, 1746–1818) 'substituted a geometrical method, which the commandant at first refused even to look at, so short was the time in which it could be practised; when once examined it was received with avidity.' Monge developed these methods further and thus created his Descriptive Geometry, the art of representing in two dimensions geometrical objects which are of three dimensions—'a problem[2] which Monge usually

[1] Thus probably originating a decimal system of numeration. It is interesting to observe in what a variety of ways (depending on the human anatomy) the decimal method may have originated and probably *did* originate—other than through the fact of man possessing five fingers on each hand.

[2] Ball, *Short account of the History of Mathematics*, p. 397.

solved by the aid of two diagrams, one being the plan and the other the elevation.'

The practical value of this method can scarcely be over-estimated, and its further development by Monge had far-reaching effects upon mathematical science itself. Here we have a new and distinct branch of science springing directly from the occupation of war, on its engineering side.

b. The Scientific impulse or factor.

The second stimulus to the development of mathematics is the impulse in humanity to systematize, and to understand natural and social facts. This we may call the *scientific* impulse or factor.

As an example, we may point to the enormous development of mathematical knowledge due to astronomy in the last four or five thousand years, and to physical science in the last three hundred.[1]

These two first factors, looking to their place of origin, viz. in stimuli received from the world around us, stimuli that come to us as sensations (i. e. through the senses), we may justly term the *external* factors in the development of mathematics.

c. The Aesthetic impulse or factor.

This is the impulse to thought, due to the vivid interest the human mind feels in its own formal creations, and the desire aesthetically to perfect these formal creations, *under necessary laws of development arising from the very nature of the mathematical stuff itself*: in brief, *the impulse to study mathematics for its own sake.* We may classify this impulse with those felt by the painter, the musician, or any other artist, and call it the artistic or aesthetic impulse in the development of mathematics. In striking contrast (speaking broadly[2]) to the two former

[1] Sturm (1803–55, a native of Geneva, Switzerland) tells us that his theorem determining the number and situation of roots of an equation comprised between given limits 'stared him in the face in the midst of some mechanical investigations connected with the motion of a compound pendulum' (Cajori, *History of Mathematics*, p. 330).

[2] How closely the *aesthetic* impulse is related to and in part coincident with the previously mentioned *scientific*, I do not here attempt to discuss. Nor do I propose to enter into the subtle question of the dependence of mathematics upon *time* as well as *space* intuitions and experience.

impulses, this has its *immediate* origin within the mind itself, and, directly at least, is largely independent of stimuli arising from actual sensation. It may therefore relatively to the others be called an *internal* factor in the history of mathematical development.

But here it is of importance to observe that this factor necessarily appears late in Mathematical history, as its very existence clearly depends upon the previous possession by mankind of mathematical experience in which one *can* be interested in this particular way. Moreover, one may add that it is an impulse whose constructive creations are rapidly exhausted—there appears ample historical evidence and warrant for this statement—unless the methods and conceptions of pure analysis with which it deals constantly revert to what has been happily termed '*the fountain of direct geometrical intuition*', i. e. to sense-perception. The first clear instance in history of the existence of this third factor is given by the Greek school, many of whom zealously cultivated mathematics in this artistic spirit: though I would add that it is easy to over-emphasize this characteristic of Greek mathematical thought, seeing that some of the very greatest Greek mathematicians were men engaged in practical affairs which constantly stimulated to mathematical research. This is so of the greatest of them all—Archimedes, equally famous as practical engineer and as pure mathematician. The combination of the 'practical', the 'scientific', and the 'aesthetic' impulses was exhibited in a rare degree in the life of Helmholtz. 'Thus it happened,' he says (in an autobiographical sketch, translated by Dr. Atkinson), 'that I entered upon that special line of study to which I have subsequently adhered, and which, in the conditions I have mentioned, grew into an absorbing impulse, amounting even to a passion. This impulse to dominate the actual world by acquiring an understanding of it, or what, I think, is only another expression for the same thing, to discover the causal connexion of phenomena, has guided me through my whole life, and the strength of this impulse is possibly the reason why I found no satisfaction in apparent solutions of problems so long as I felt there were still obscure points in them.' In these last concluding words there appears more than a touch of the 'artistic' impulse.

3. *Educational Import of the above facts: the Race and the Individual.*

With an eye to the educational value of these facts, I repeat again that mathematical knowledge has gradually grown in direct response to certain vividly-felt needs of our ancestors, and is still rapidly growing under similar impulses in our own generation. Let not the teacher and pupil harbour the vague and deadening idea that mathematics is some artificial activity of human life, mysterious in origin and appealing to but a few.

I have summarily reviewed the impulses, external and internal, that have ever been urging humanity to develop mathematical knowledge. I have now to consider the nature of this knowledge, the kind of mental processes by which its development has been successfully brought about, and, finally, the bearing of these facts upon education. For there is every reason to believe, looking to the practically unchanged constitution of the human mind for at least several thousands of years back, that those factors which have been throughout essential to the growth of mathematical knowledge in the minds of our ancestors must be closely similar to, if not actually identical in kind with, the main factors that underlie efficient mathematical education in kindergarten, school, and college. We have, indeed, already indirectly alighted upon *one* factor indispensable to efficient education: one may perhaps claim it as the chief factor, for upon its existence all others appear ultimately to depend. It is this—*That the pupil should feel vividly the need for any particular piece of mathematical knowledge that is in question: whether this felt need springs from 'practical' demands occurring in intelligently-conducted practical measurements in mensuration, woodwork, metal work, art, &c.*[1]) *or from 'scientific' demands* (*occurring in the study of mechanical*

[1] The best manual teachers are already recognizing measurements in connexion with training in wood and metal work as an excellent instrument for the lucidation of fractions. Cf. 'Doubtless the Romans unconsciously hit upon a fine pedagogical idea in their concrete treatment of fractions. Roman boys learned fractions in connexion with money, weights, and measures. We conjecture that to them fractions meant more than was conveyed by the definition "broken money", given in old English Arithmetics.' (Cajori, *A History of Elementary Mathematics*, p. 41.)

and physical experimental science, geography, &c.) or—much more rarely—from the 'artistic' impulse.

This last impulse we find in young Pascal, who, at the age of twelve, quite unaided, evolved a whole system of geometry by drawing figures with charcoal upon a tiled pavement; and in the famous calculator and engineer Bidder, who, also unaided, in quite early youth discovered many fundamental properties of numbers (amongst others, the existence, meaning, and value of the coefficients of expansion in the binominal theorem for a positive integral index). Indeed, this factor—*a vividly-felt need, arising from one or a combination of the above impulses*—is indispensable for evoking interest, and maintaining attention—without which no efficient education can be given.

CHAPTER XVI

DIAGRAMMATIC SKETCH OF THE DEVELOPMENT OF MATHEMATICAL EXPERIENCE

The Diagram: its Objects (see Frontispiece).

I now proceed to a brief sketch of the rise and growth of mathematical knowledge.

In the diagram forming the frontispiece I have attempted to symbolize this development graphically so as to exhibit:—

1. The kind of *external* stimulus to which its growth has been and is still due—in fine, its *concrete* basis in matters of fact.

2. The *kind* and *quantity* of the world's mathematical knowledge (omitting Chinese) at any given era of its history with which we are reasonably well acquainted. The interpretation of the shading is stated on the page opposite the diagram, and has already been sufficiently elaborated in preceding chapters. (See Chaps. V, VI.)

Of course this quantitative symbolization is but of the most loosely approximate[1] kind, purposing merely to convey the strength of the feeling produced on one's mind in calm contemplation of the contributions to Mathematics made by any given race. The outcome of sociological inquiry into the origin and development of human occupations is still so obscure that none but the broadest features of the developments appear to have substantial validity, and even these, as here utilized, are to be interpreted in a general rather than a literal sense. The field of research in this fascinating, complex, and highly valuable branch of history

[1] This and other limitations (e.g. see p. 245) were apparently insufficiently appreciated by a distinguished critic who objected to the *linear* form of the developmental diagram. See also p. 369.

will, it is hoped, gradually attract more earnest students. The Time-axis runs through the centre of the chart. From 3000 B.C.[1] onwards, the scale is roughly two inches for every thousand years: antecedent to this the time-scale is purely symbolical, and one may imagine the chart extended towards the narrower end to an indefinite extent, ever growing narrower and whiter.

The writer may perhaps be permitted to say that his chart, once constructed, has proved itself so fruitfully suggestive to himself that he ventures to hope that patient and detailed study of it may prove not less fruitful to others.

The general ideas suggested appear not inapplicable to other branches both of knowledge and of education. But judgement on this must be left to others; better qualified to judge as to their own particular studies.

The External Factors in the Development.

(i) *Physical Environment.*

Here, to some extent, repetition is almost unavoidable. To start with, we have the factors arising from man's very existence as an animal with a body of his own—with size, shape, and mass. In this aspect man may be justly termed a *mechanical mass*, a fact which, obviously, from the very first moment of his existence as a separate entity, is of fundamental importance in the evolution of mental processes concerned with spacial and mechanical phenomena. Even before our ancestors could be justly described as 'primitive men' their consciousness must have been well stored with fundamental mathematical material. Even the animal—monkey, small boy, or what not—that can throw a well-aimed stone has truly a grasp of *applied* mathematics that is not to be trifled with.

The close dependence, indeed, of original number-concepts on the anatomy of the human body is seen in the fact that, with a few isolated exceptions, all nations are found to possess 5, 10, or 20, as the basis of their system of counting—5 when the fingers of *one* hand is taken, 10 when the

[1] The above was written twenty years ago. Recent researches now show that the Archaean period should be magnified to at least double its length as shown on the diagram.

fingers of both hands are taken, and 20 when both fingers and toes are taken.

The dependence of simple *geometrical* concepts on the same source—the human body—is well illustrated in the following diagram :—

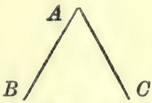


FIG. 77.

The side AB or AC of an angle is called	*arm* or *leg* in English			
	jambe in French, signifying also human *leg*			
	crus in Latin,	,,	,,	,,
	σκέλος in Greek,	,,	,,	,,
	bâhu in Hindu,	,,	,,	*arm*
	kou in Chinese,	,,	,,	*hips.*

(ii) *Primary and Derivative Occupations.*

Next we have those factors arising from the struggle of primitive man (including, of course, also the woman) with Nature for shelter, clothing, food, and so on, and of the earliest types of civilization in the evolution of such *primary and derivative* occupations as that of the shepherd, hunter, fisher, miner, forester, peasant, sailor, trader, smith, artist, carpenter, rising at length into the highly specialized and professional occupations of the warrior, the scribe, the musician, the priest, the teacher, the engineer and architect, the doctor, the lawyer, the accountant, the civil servant, the administrator, wherein the mathematical and other sciences doubtless made increasingly rapid progress. Both sexes have contributed to the development, though the rôle of the male in modern times has been predominant. The origin of those instruments for the measure of number, space, time, mass, and value—what corresponded, in fine, to the Abacus, Foot-rule, Sun-dial, Balance, and Money—is still wrapped in obscurity. The use of such devices marks an immense advance in the progress of Mathematics : we have evidence of their existence for at least 7,000 years. Note the value and complexity of these inventions : it is a great advance to develop *simple* units, but it is a vastly

greater to develop *collective* units, such as all these devices imply.

Further, all fundamental truths must be *felt* and *instinctively acted upon* with some degree of reliance long before they can get definite formulation and abstract demonstration. Such was pre-eminently the spirit of this early era, for it was, on the great mass of material, developed in such a *practical*, *unconscious*, and *instinctive* way by the Archaean civilizations, and handed on to the Greeks, that the latter built their refined and powerful mathematical structure.

This principle has wide application to mathematical and other education.

The influence of physical environment and occupation on the growth of Mathematics is well shown in Egypt. Herodotus even goes the length of believing and stating that the very *origin* of geometry was due to the necessity of surveying the Nile districts on account of the river's periodic overflow, which obliterated all traces of private property. We may safely demur to this as going too far, without denying the great influence of such a factor on the progress of a geometry already otherwise brought into existence.

The influence of 'occupation', especially of the professional occupations, upon the evolution of Mathematics, though perhaps at its maximum in ancient times, is also a potent factor in all times. This is particularly the case with engineering and shipbuilding. This relation of occupation to science I have already touched upon.

(iii) *Influence of Astronomy, Mechanics, and Physics.*

The influence of Astronomy (itself springing from Astrology,[1] and this in turn closely allied with Religion and Statecraft) and of the Physical Sciences, including Mechanics, I have already noted. This is so well known and understood, in comparison with the other factors, as, indeed, it has been and is the most considerable of all, that, for the moment, I pass on, with the remark that, as they have formed the most influential factor in history, *so should*

[1] Originally, Astrology was an Art largely employed for fixing the time of agricultural operations, by observation of the stars' movements, &c.

these sciences have the preponderating weight in the correlation of Mathematics with other studies in education.

(iv) *Influence of Chemical, Biological, and Sociological Studies.*

Lastly, and as yet in the main comparatively insignificant, but possibly soon to be of high value, there is the influence (not symbolized in the diagram) of Chemical, Biological, and, above all, of Sociological phenomena through Statistical Science on the growth of Mathematics ; Statistical Science itself being, as a branch of probability, fruitful in the discovery, from Pascal and Fermat onwards, of purely mathematical truths.

The Internal Factors, or Stimuli.

So much for the external factors depicted in the diagram. Those more subtle internal factors, due to the fascination of the study in itself, regarded as a truly aesthetic product of the human mind, have necessarily been omitted from the diagram. To evaluate their precise strength, generally even to get evidence of their very existence in any degree of purity, is obviously a difficult task. Nevertheless, from the nature and history of the aesthetic impulses in other branches of human activity, and from scattered hints here and there in the early history of Mathematics, we are confident that this influence has operated, with more or less effectiveness, from the earliest recorded times. We find it comparatively strongly developed among Hindus, Arabians, and Greeks, and in modern times among many European nations this factor often reaches the height and influence of a veritable passion.

Educationally, this internal factor has its proper place as a stimulus, often of considerable strength, to the young mind, in co-operation and combination with the others.

Finally, to avoid misconception, it must be repeated that although certain factors or stimuli have shown, in comparison with the others, their maximum effect at certain periods with certain nations and races—most of which are portrayed in the chart—at the same time *all of these* factors (external and internal) have also continued to act when once started, and to all appearances will ever act upon

mathematical development, though with relatively varying intensities.

The Quality and Quantity of the Race's Mathematical Knowledge at various Epochs.

Remembering the interpretation of the shading on the chart, and speaking broadly—by taking a section of the chart perpendicular to the time-axis, we may estimate :—

1. The *quantity* of Mathematical knowledge in the possession of the race at any period of history.

2. The *quality* of this knowledge with respect to its concreteness or abstractness, its sensuous or intellectual character, its empirical or scientific character.

The *quantity* is represented, extremely roughly of course, by the total depth of the section-line, and the quality by the *relative* depths of its shaded and unshaded portions. Taking, as example, a section of *Greek* civilization at about 500 B.C., we interpret the equality of depths of the shaded and unshaded portions and the slope of the line dividing these two portions to mean that, at this period, Greek Mathematics is rapidly passing from a predominantly concrete stage to a predominantly abstract stage—a truly critical point in the history of Mathematics.

I do not propose here to treat of the historical development *in detail* of mathematical knowledge : a list of works, English and foreign, will be found in the last chapter by the aid of which the reader unacquainted with that history, and using the chart as a guide for the marking of the main advances, may become familiar with its details and the lives of great mathematical thinkers. Subsequently I hope to treat, with some detail, of certain characteristics of the evolution of this science which appear to me either to be obscurely grasped, misrepresented, or even entirely neglected by historians.

Broadly speaking, the nature of Mathematics and the nature of its growth in the race appear to be justly described in the following statements :—

I. Its two fundamental elements may be combined in the most varied proportions.

II. It has passed from a highly predominant concrete stage (from almost pure sense-perception) continuously

through intervening varieties to a highly predominant abstract stage (to almost pure conceptual construction).

III. Both elements appear, from almost vanishingly small beginnings, to increase without limit in absolute quantity, but the conceptual element ultimately increases more rapidly than the perceptual. Indeed, it almost appears as if the ratio of conceptual to perceptual, itself, tends to increase without limit.

The main evidence for Statement III is based on (i) the vast range of natural phenomena—sense-perceptions—that have received mathematical expression, and (ii) the still vaster range of highly symbolical thought (almost purely conceptual) known as the Higher Arithmetic, the Theory of Functions of the Complex Variable, and, above all, of Universal Algebra, that has appeared in the last hundred years in so remarkable a manner that it may almost be said to have suggested the very lines of its own development. Hence this last and immense contribution may be taken, perhaps, as in great measure due to stimuli springing from the very heart of Mathematics itself.

The Kinds of Evidence for Mathematical Truth.

The kinds of evidence or proof that have been historically offered—or, in fact, which, apparently, ever *could* be offered—in support of geometrical statements, I propose to classify thus :—

I. *Experimental* evidence or proof.
II. *Intuitional* evidence or proof.
III. *Scientific* evidence or proof.

I have previously discussed somewhat fully the relations and characteristics of these three kinds of evidence. Here I confine myself to the following bare remarks :—

Experimental evidence can establish only particular truths, but may suggest general truths : the predominant mental activity in its use is sense-perception ; and this mode of evidence characterizes much of the mathematical knowledge of primitive man.

Intuitional evidence can establish general or universal truth, and may suggest the scientific ideal of evidence : the mental activity employed in this kind of proof is a tolerably equal combination of sense-perception and conception. It is characteristic of Archaean mathematics and of much of

early Hindu and Greek. These two—intuition and experiment—are *par excellence* the tools of discovery.

Scientific evidence implies an inter-connected systematization of already discovered general truths: the mental activity predominating is abstract conception. It characterizes much of later Greek mathematics and most of modern European mathematics—although, as an absolute ideal, it never has been and probably never will be fully attained.

The actual order of development has been almost invariably from I through II to III. The path of *discovery* in mathematics is seldom or never along a strictly scientific road: this comes later.

Illustrations.

By studying the annexed Figs. 78, 79, 80, showing the history of the development of the famous Pythagorean proposition (Euclid I. 47), the reader will have an excellent illustration of the typical mode in which mathematical truths have been discovered, developed, and 'proved'.

At least as early as 2000 B.C. the Egyptians were in possession of the fact that, if the sides of a triangle are as 3 : 4 : 5 nearly, then the largest angle is nearly a right angle, and that the more accurately these ratios were actually subsistent in the concrete triangle employed, the more nearly did the largest angle appear to equal a right angle (see Fig. 78). So far as we are aware, their use of the truth was limited to this very concrete significance. The test, in any doubted case, was experimental, by actual measurement. To a *general* proof establishing the rigorous equality to a right angle *if* the sides could be rigorously made to satisfy the exact ratios, this nation does not appear to have risen. At least there is no evidence of such, but much evidence against it. Observe here, first, that measurement (i. e. experimental evidence) can establish only the *approximate* truth in the *particular case actually measured*: it is a *generalization* directly proceeding from the particular statement, 'This particular triangle lying here, whose sides are 3 ft., 4 ft., 5 ft. respectively, has such and such a numerical property,' to the statement, '*All triangles whose sides are as* 3 : 4 : 5 have such and such a numerical property.' It is

I.—Egypt, about 2000 B.C.

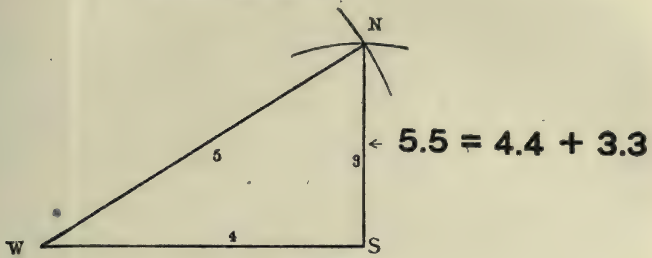


Fig. 78.

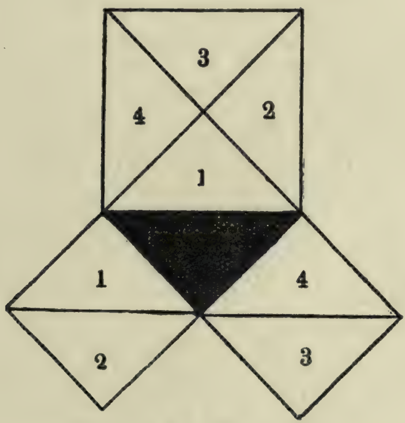


Fig. 79.

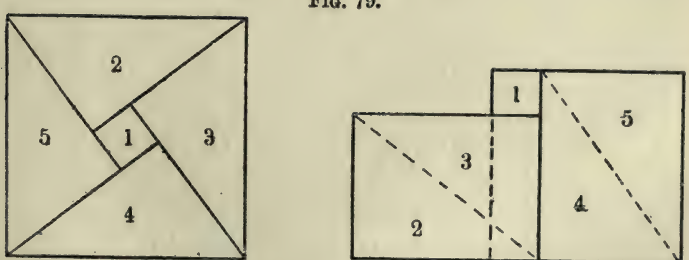


Fig. 80.

possible and probable that intuition of similar figures would enable the Egyptians to discover such a generalization—e.g. if the triangles were arranged thus:—

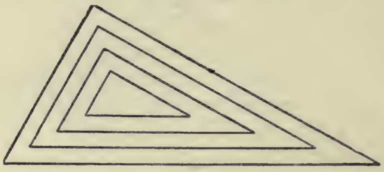


FIG. 81.

i.e. were *optically* as well as geometrically similar.

This truth was very valuable in orienting their temples and other buildings. A north and south line being obtained by star-observation, as from N to S (see Fig. 78): then a rope being stretched from N to S, whose length is 3, a point W due west of S can be obtained by the use of two other ropes of lengths 5 and 4, as indicated.

Passing on to Greece, about 550 B.C., we find the Greeks in possession of a simple *intuitional* proof of the particular case of I. 47—but itself, observe, a *generalization*—when the triangle is isosceles (see Fig. 79), and also of the general proposition itself by *a process of dissection* (appealing also to simple intuition) *applicable to all right-angled triangles* (see Fig. 80). This mode of dissection is credited to the Pythagorean school, but it was familiar to the Hindu mathematicians about two hundred years earlier, i.e. about 800 B.C., and to the Chinese about 1200 B.C.

Finally, we have the well-known '*scientific*' proof of Euclid, in which, upon all the paraphernalia of axioms, postulates, and definitions, there is gradually raised a number of mutually dependent and systematized truths ultimately leading to the proof of I. 47. That the procedure, even in this final case, is far from rigorously scientific has been repeatedly noted: one may indeed say that there repeatedly happens that which occurs in the very first proposition of the First Book. Here an assumption is made (that the two

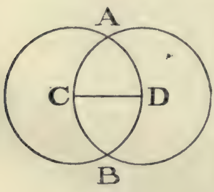


FIG. 82.

circles described with C, D respectively as centres, and with radius equal to CD itself, will *cut each other*) which does not follow from the definition of a circle alone without further appeal to the properties of flat space *as given by sense-perception*; a new appeal is consequently made to intuition. *The whole Euclidian treatment would be nugatory were this appeal to sense-perception not being constantly made.* Well may we say 'Expel Nature with a pitchfork: yet she ever returns.' By the way, remark that a copious use of postulates (particularly when unconsciously made—as is generally the case with the naïve mind of the young and of early civilizations) need in no wise diminish the logical training involved. It is for the above total failure of the Greek geometry to exclude sense-perception during the course of its proofs, and for its failure to evolve analytical methods and results, that I have ventured, though with great hesitation, to class the Greek Mathematics, on the whole, *behind* the Hindu. This latter race also appealed to sense-perception equally, and even more unconsciously, in pure geometry, but, if we may accept the usual view, in analysis (arithmetic, algebra and trigonometry) they far outstripped the Greek school (if we except Archimedes, whose achievements are certainly puzzling in their astonishing range).

I venture to think that this strongly intuitional character of Greek geometry has not been sufficiently recognized: I revert to it again shortly. As, for my present purpose, it is of great importance to make clear the broad distinctions between the three fundamental species of evidence for mathematical statements, I append another set of illustrations.

Further Illustrations of the Kinds of Mathematical Proof.

The history of the most famous and radical property of rectilinear closed figures is wrapped in obscurity. Nevertheless, I venture, simply for the purpose of illustration, to describe its probable mode of evolution so far as I have succeeded in grasping the essential spirit of the historical development of mathematical truth.

In the first place, among the ancient races, the purely *experimental* evidence of *actual measurement* must, early, have both suggested the fact and evidenced the truth (within

experimental limits of error) that the sum of the angles of any actually drawn triangle amounts to two right angles.

But, as I have already insisted, evidence of this kind, derived from actual measurement, can establish neither rigorous equality nor universality.

Many experimental approximations of this kind were handed over from Egyptian mathematics. Note (to follow out our hypothetical illustration) again in what a striking way the subtle Greek mind attempted to transform these isolated pieces of empirical or experimental fact into a con-

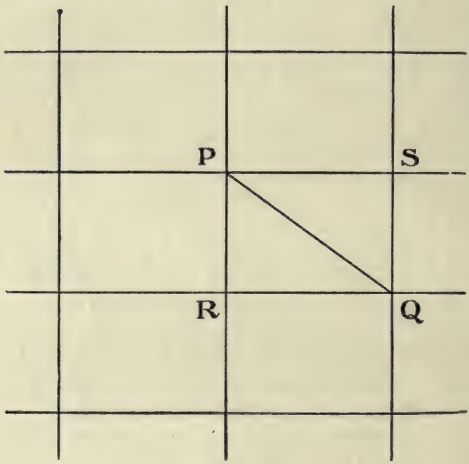


FIG. 83.

nected system of universal truths and thus created a precedent and model of inestimable value to future races in the development of science.

The observation of tessellated pavements (Fig. 83) would, to the inquiring geometrical genius, lead to the truths (i) that every rectangle can be dissected into two precisely identical right-angled triangles, as *PRQ* and *QSP* in the above figure, and, conversely, that two such triangles can be constructed into a rectangle; (ii) that, as the sum of the angles of the rectangle is four right angles, therefore the sum of the angles of its half, viz. a right-angled triangle, must

be two right angles; (iii) therefore that, in every right-angled triangle the sum of the acute angles is one right angle. But a further observation of triangles would convince the observer that *every* triangle, whatever its shape, is decomposable into two right-angled triangles, by the simple expedient of folding it in a certain way, or, otherwise describing the result, of dropping a perpendicular on to the longest side from the vertex opposite, as in Δ *ABC*, Fig. 84, where *AD* the perpendicular divides the triangle into two right-angled triangles, Δ *ABD* and Δ *ACD*. This fundamental property of a triangle is not sufficiently *consciously* insisted upon in teaching, though the basis of the elegant science of trigonometry.

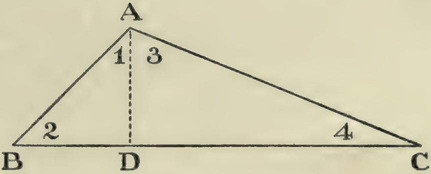


FIG. 84.

Now apply the general truth just gained by observation of the tessellated pavement to each of these triangles, ΔABD and ΔACD. Each is a right-angled triangle: therefore, $\angle 1 + \angle 2 = 1$ right angle: also $\angle 3 + \angle 4 = 1$ right angle, $\therefore$ the four angles 1, 2, 3, 4 equal together two right angles. And, $\angle 1 + \angle 3$ making together $\angle A$, we have the result that, *in every kind of triangle the sum of the angles equals two right angles.*

Now here we immediately admit the *generality* of the truth: but, although we have appealed repeatedly to sense-perception, we have also relied on the *generalizing* impulse of the mind, and on an element of mental activity which we usually call 'intuition'—i.e. apparently *direct* perception of a general truth. In fine, we have here *intuitive* evidence, implying, in Mathematics, the capacity to draw *immediate and apparently universal* conclusions *from generalized constructions* such as we had above.

Such was the nature of the evidence by which the early Greek geometers converted the empirical results of Archaean mathematics into the rudiments of a universal geometrical science. Note that, to the mental activity involved in the purely experimental stage, an *additional intensity of thought*

is added. The evidence is more difficult to master than the purely experimental. Euclid's predecessors in this way gradually imported the element of generality into the Egyptian calculations. But the truths thus reached remained for long a mere collection of more or less isolated facts, each resting on some particular intuition (the above, for instance, implies the postulates :—such figures as rectangles exist : each rectangle is two identical right-angled triangles, and conversely : each triangle is two right-angled triangles, and so on), and not in any high degree dependent on other similarly proved truths for its evidence.

Then came Archytas, Euclid, and others, who, stimulated by the criticism of logical thinkers of the calibre of Plato and Aristotle, boldly conceived the vast ideal of welding the enormous mass of comparatively isolated truths discovered by their predecessors into an inter-connected systematic whole. In such a whole each truth was to rest on its neighbouring forerunner, and the whole system, in one long chain of rigorous deduction, be securely founded on the *least* number of fundamental truths—postulates, axioms, and definitions—to which they could possibly be reduced.

Now when a system of truths can thus be connected so that they appear to the searching criticism of the age to 'stand or fall together', the grand title of *science* is bestowed upon it. This stage of mathematics, then, we may call *science*, and the highly complex kind of proof or evidence it implies is *scientific*. But this complete interdependence and perfect mutual support is an unrealizable, though none the less valuable ideal, if we may judge by the whole experience of mathematical history. In my opinion it is therefore safer to say that a body of knowledge is the more scientific the nearer it approaches this ideal and the vaster that body is. Each generation in this respect is constantly improving on its predecessor, as refinements of abstract invention increase in subtlety and sharpness of observation advances with it. Proof is continually being perfected, and there is no end to the perfection: its maturity is strictly relative and proportioned to the intellectual grasp of the nation from which it springs, and, within that nation, to the intellectual maturity of the individual.

We have seen that Euclid and his fellow workers did

not attain, and could not have attained, perfect success in their ambitious undertaking. Although the result was a marvel of achievement for centuries, it is really riddled with sense-perception assumptions, it appeals to intuition, from beginning to end. The *rigorous* execution of the undertaking has already occupied mathematicians for over 2,000 years, and (to prophesy by the light of experience and all the deepest human thought on the nature of *rigorous* scientific proof) is likely to occupy them 2,000 years longer! Sense-perception and intuition, from the very nature of knowledge and its conditions of development, can never be wholly banished from mathematical science by being relegated, in the shape of axioms and postulates, to the beginnings of the science; for they form the very life-blood of mathematical discovery.

In this last kind of evidence, then—the scientific—the element of mental activity that is predominant is the conceptual or abstract. But, as we have shown, it is a misapprehension to call the Euclidian treatment *rigorously* abstract and logical. Were it really so—despite our carelessness in educational matters—Euclid would long ere this have been banished from the school as totally unfit for young, immature minds. Yet it is not merely just but important to admit, now the reformers have won the day, that many excellent teachers achieved—and still continue to achieve—distinct success in utilizing Euclid effectively both for discipline and use. All geometry partakes of the nature of an *experimental* science (such as Physics) as well as of a deductive abstract science (of which an ideal algebra would be the fittest type). Nevertheless, although this sense-basis of Euclid has sometimes happily saved the situation, educationally, from being a farce, it is, on the other hand, clear that the whole idea of a systematized interconnexion of truths resulting in a more or less perfected science is beyond the capacity of average boys and girls. Its very existence presupposes the pre-existence of a large body of geometrical facts arrived at by the path of experiment and intuition. The best path to be followed by the pupil is clearly *through the experimental and intuitional* stages successively onwards to the *scientific* stage: what the race has never been able to accomplish, it is unsound to expect of the child, boy and girl. Only by the path of very

gradual creation from the first and second species of evidence, wherein sense-perception relatively decreases in quantity while conception and abstraction gradually replaces it, is it possible for either race or individual to approach the summit of knowledge—rigorous scientific truth.

The stage of education at which it is appropriate, if at all, to undertake the conversion of the previously gained experimental and intuitional mathematical knowledge into a system reasonably scientific and rigorous will clearly depend upon its grade and specific object. In schools where the curriculum is designed to embrace a course of 4, 5, or 6 years beyond the age of 12, there should be ample time to systematize the mathematical studies in this scientific way, though of course on modern lines—preserving the Euclidian spirit though incorporating modern matter.

CHAPTER XVII

EDUCATIONAL PRINCIPLES : THE EVOLUTION OF MATHEMATICAL KNOWLEDGE IN THE INDIVIDUAL

Two Fundamental Conditions of Efficient Teaching as regards the Matter.

THE *potential* value of mathematical training, for life-use, for culture, and for mental discipline, is never seriously contested. In the following remarks we take this for granted.

But as to the most efficient methods of transforming this admitted potentiality into actuality, opinions and practices widely differ. Educational experience of different ages and different countries appears to warrant the statement of two fundamental conditions, the fulfilment of which is necessary and sufficient for the effectiveness of mathematical training—so far as the *matter* is concerned.

I. *The particular mathematical experience which forms the material of the educational process must, at every stage, both in quantity and quality, be appropriate to the present capacity of the individual who is expected to assimilate it.*

II. *The correlation between the different branches of pure mathematics themselves, and between these latter and the manifold applications of mathematics must be natural, closely interdependent, interesting, and continuous throughout.*

The meaning, here, of this much used and much abused word 'natural' will plainly appear in the sequel. Meanwhile, I simply remark that any correlation is regarded as *natural* which has *historically* existed in the development of the science (see Diagram in frontispiece).

The attitudes of the pupil and teacher I shall deal with subsequently : here I deal with the *material* of the mathematical teaching. The question now is, How are these two fundamental conditions to be realized ?

Parallelism between Race and Individual.

Slowly in the last two thousand years, more rapidly in the last hundred, there has dawned a principle which, applied with common sense and with limitations imposed justly by personal experience, bids fair to become one of the central truths of education.

For my present purpose I venture to state it formally thus :—

The path of most effective development of knowledge and power in the individual, coincides, in broad outline, with the path historically traversed by the race in developing that particular kind of knowledge and power.

Granting this—and all the best winnowed experience of education goes to establish its truth, so that the teacher may justly now regard it not as a doctrinaire statement *but as a broad and fair summary of experience itself*—our educational task is much simplified. The teacher has simply (not easily : education, if efficient, is seldom easy) to conduct his experiments on the broad lines definitely suggested by the racial development of mathematical experience. Fortunately, we are reasonably familiar with this development : in broad outline I have striven to trace it in the preceding pages. For educational purposes I have now to re-translate the fundamental laws of this racial development into terms of individual development, in harmony with the two prime conditions which have been definitely formulated above.

I am aware that the validity of this 'culture-epoch' principle, as it has been called, has been vigorously attacked : but never, I believe, by any competent teacher who has fulfilled these conditions :

(1) *Had familiar knowledge of the spirit and the main outlines and details of the historical development of the subject taught.*

(2) *Experimented, for a considerable number of years, in testing the value of the principle by actual application to his own pupils and classes over wide limits of age.*

The value and general truth of the principle we hear on all sides—from such different authorities and teachers as Boole for mathematical teaching, Mach for physical and mechanical, Michael Foster for physiological, Miers for crystallographical—not to mention still greater and older authorities, such as Plato, Pestalozzi, Goethe, Froebel, Herbart, &c.

The Historical Chart re-interpreted in terms of Individual Development: Central Principles.

Referring to the Historical Chart, I would ask the teacher to replace the *racial* lines of capitals on the left by the corresponding lines of capitals on the right, noting the kind of substitutions thereby made, and the chart at once becomes representative of the development of mathematical knowledge in the *Individual*.

Thus, for each age of the individual life—infancy, childhood, school, college—may be selected from the racial history the most appropriate form in which mathematical experience can be assimilated. Thus the capacity of the infant and early childhood is comparable with the capacity of animal consciousness and primitive man. The mathematics suitable to later childhood and boyhood (and, of course, girlhood) is comparable with Archaean mathematics passing on through Greek and Hindu to mediaeval European mathematics; while the student is become sufficiently mature to begin the assimilation of modern and highly abstract European thought. The filling in of details must necessarily be left to the individual teacher, and also, within some such broadly marked limits, the precise order of the marshalling of the material for each age. For, though, *on the whole*, mathematical development has gone forward, yet there have been lapses from advances already made. Witness the practical world-loss of much valuable Hindu thought, and, for long centuries, the neglect of Greek thought: witness the world-loss of the invention by the Babylonians of the Zero, until re-invented by the Hindus,[1] passed on by them to the Arabs,[1] and by these to Europe.

Moreover, many blunders and false starts and false principles have marked the whole course of development. In a phrase, rivers have their backwaters. But it is precisely the teacher's function to avoid such racial mistakes, to take *short cuts* ultimately discovered, and to guide the young along the road *ultimately* found most accessible with such halts and retracings—returns up side-cuts—as the mental peculiarities of the pupils demand.

All this, the practical realization of the *spirit* of the principle, is to be wisely left to the mathematical teacher, familiar

[1] Some doubt has been recently thrown on these two last-mentioned facts.

with the history of mathematical science and with the particular limitations of his pupils and himself.

As an instance of a stage which experience shows it to be *unnecessary* to traverse in the application of this principle may be mentioned the addition stage in the *notation* for numbers. The diagram below indicates the way in which the Egyptians symbolized numbers, in their final or hieroglyphic stage. The diagram is self-explanatory, so I merely ask the reader to study its contents carefully before passing on.

EARLY EGYPTIAN NUMBER SYMBOLS (HIEROGLYPHICS).

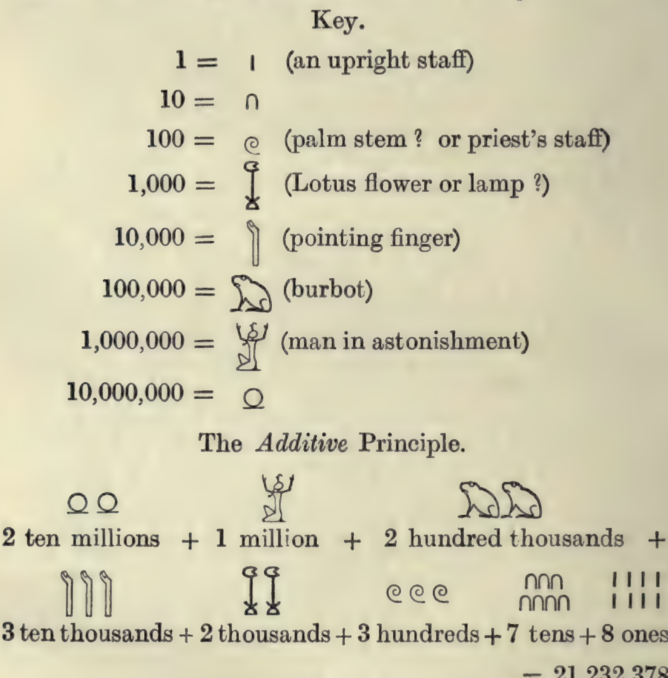


FIG. 85.

N.B.—To save time in *reading*, the Egyptians placed 4 units of the *same* kind, at most, together in one line

(e.g. note 78 above). We still use the same method, with the same limitation to four, in many businesses where objects are counted by 'chalking' them up one at a time.

In *principle*, with concrete objects of symbolization though not with written symbols, the young child will find it advantageous, in general, to pass through this stage, as when bundles of single sticks are tied together to symbolize a higher unit, and then replaced by a single object of a different kind: and so on. But it would clearly be loss of valuable time to commit to memory written symbols (as in the above hieroglyphics) for these different units which would afterwards require to be forgotten and replaced by the much more perfect zero and place system of symbolic numeration.

Nor—again reverting to Egyptian mathematics—was their system of *unit-fractions*, nor of the solution of linear equations, one we should desire to imitate, cumbrous in the extreme as each was.

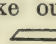
One of the earliest known Equations in Algebra.

(From an Egyptian Papyrus, about 2,000 B.C.)

$x = 14 + \frac{1}{4} + \frac{1}{97} + \frac{1}{56} + \frac{1}{679} + \frac{1}{776} + \frac{1}{194} + \frac{1}{388}$ is the 'root'.

$x \quad (\frac{2}{3} \quad + \quad \frac{1}{2} + \frac{1}{7} \quad + \quad 1) \quad = \quad 33.$

Fig. 86.

In the above diagram is given the Egyptian symbolism for the equation $x\,(\frac{2}{3}+\frac{1}{2}+\frac{1}{7}+1) = 33$ immediately above it. At this period the Egyptian mathematicians employed none but unit-fractions (i.e. fractions whose numerators are unity), with the single exception of $\frac{2}{3}$, for which the symbol was . The reader will, by the relative positions of the Egyptian and modern symbols, with little trouble make out the nature of the correspondence, if we add that  $= \frac{1}{2}$:  $= \frac{1}{7}$: and ∩ = 10.

The Abacus and Zero

But, on the other hand, the evolution of the zero and place-value, and of the index—with many other inventions

in mathematical symbolism that might be mentioned—might well be utilized for teaching purposes. These I shall refer to shortly, and they form examples of occasions when even the *details* of the historical development are highly valuable to the teacher in practical work, though I cannot too strongly insist on the view that it is mainly by the *spirit* of history we must be guided if we would fruitfully apply the historical principle of parallelism above enunciated.

I even venture to maintain that *wherever an admittedly genuine advance has been made in the teaching of mathematics, such advance (whether the reformer has been conscious of the fact or not) has invariably been of a nature fully harmonious with this our central principle of parallelism.* This, if true—and I have repeatedly tested its truth—is in itself solid evidence of the value and truth of the principle. Perhaps I may here be permitted in support of the main view above to repeat an illustration already used.

Teachers of arithmetic are familiar with the difficulty most children have in mastering the use of the principle of place-value and of the zero.

Some years ago a teacher in the west of England (a lady, whose name I do not know) discovered that the passage from the abacus to the perfected written symbolism could be made vastly easier by the intermediacy of a kind of *paper-abacus,* in which *columns were substituted for the rods* when numbers were substituted for the beads. The accompanying diagrams will show the spirit of the invention :—

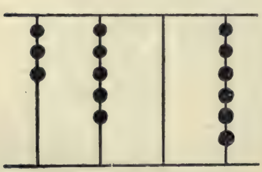


FIG. 87.

Stage I
Wooden Abacus with beads.

Thousands	Hundreds	Tens	Units
3	5		6

FIG. 88.

Stage II
Graphic Abacus with written figures and columns.

3506

FIG. 89.

Stage III
Fully developed written symbolism.

The introduction of Stage II, where the columns replace the rods of the original abacus, and the written symbols the beads, makes easy the final passage to the fully developed symbolism, in which, by the substitution of a symbol (zero 0) for the *blank column when on the blackboard the lines dividing the columns are wiped out*, the principle of place-value is thus introduced almost unconsciously.

Now this ingenious teacher (without apparently being aware of the fact) by this device, simply traversed with her pupils the *three main stages which this system of modern numeration has historically passed through.*

First we have the abacus (common in some form or another to almost all ancient nations) in which relative position determines the size of the unit (itself a great advance on previous simpler systems) but in which the actual individuals of that unit are tangible and visible objects (beads) and hence highly sense-perceptual. Here we have a predominancy of sense-perception.

Secondly, 'in the early middle ages the use of the abacus or reckoning-machine was common in Europe, just as to-day in China and Japan. From the abacus was invented a principle of notation which may specifically be termed the *column* principle. For certain kinds of work this method is very convenient, and it is still used in many ways for special purposes. It was employed in the early dark ages much more than at any other period before or since. The following illustrates the method :

M	C	X	I	
2	4	5	2	= 2452.
2			6	= 2006.
1		6		= 1060.
	4	5		= 450.

'With such a Graphic Abacus, Zero is unnecessary : it gives a method by which, with 9 figures only, the highest numbers can be written.' A similar series of steps was

traversed, Arneth tells us, in India, one of the original homes of the Zero.

Finally, it may be added that the word Zero itself (and its doublet 'cipher') means in Arabic *empty space*, (being related to the well-known word Sahara), indicating thus its origin as the symbol used for denoting the absence of value at a certain place in a row of figures. At this point (Stage III) we have reached a stage in which abstraction is the highly predominant factor, hence the difficulty of grasping it if the young mind is not prepared by the intermediate and original stages which are more concrete in character. I have myself noted the great ease with which children grasp and master the use of the principle of place and zero by the employment of this *column* device (Stage II above): in fact I have found that Stage I can more readily be dispensed with than Stage II.

With one child, when the columns were rubbed out or otherwise dispensed with, and the *child of itself saw that some sign for indicating a blank space (or column) was absolutely essential, a dot was suggested by the child* (which was the original Arabic symbol, for *zero*), and this was actually used for a little until it was found that its being easily rubbed out or missed over led to misinterpretations, when the full zero 0 was gratefully accepted as a more reliable symbol—especially when a little talk on the history of the zero was added: an addition I commend to other teachers. This little experience illustrates further a principle upon which I lay great emphasis, viz. introduce no new symbols until the need for them is vividly felt, and therefore the use of them genuinely welcomed and appreciated.

Decimal Notation.

My second illustration of the value even of the *detail* of mathematical history to teachers concerns the use of the decimal point. It is the custom among many good teachers, in dealing with the difficulties of the decimal point—especially in division—to place small figures over each succeeding decimal place to indicate the particular sub-division it denotes. Thus:

$$78{\cdot}69537$$

is written for a time

$$78{\cdot}\overset{1}{6}\overset{2}{9}\overset{3}{5}\overset{4}{3}\overset{5}{7}$$

the 1 reminding the pupil that the 6 denotes 6 tenths
2 ,, ,, ,, 9 ,, 9 hundredths
(i.e. second power of $\frac{1}{10}$)
3 ,, ,, ,, 5 denotes 5 thousandths
(i.e. third power of $\frac{1}{10}$),
and so on.

In essence, indeed, this is the principle of the column-device again.

Now this very device was constantly employed by the great mathematicians who invented and first used the decimal notation; occasionally, it is true, with slight modifications, e.g. using 0, 1, 2, 3, . . . in place of 1, 2, 3 . . . respectively: a very significant substitution, by the way, for the development of the theory of indices. Thus we find it almost universally in Stevinus, the reputed Dutch inventor of the decimal point and decimal fractions; also in mathematicians who followed him, until in time, when men became adept and familiar with the principle and symbol, the device was gradually dropped. We may not, indeed, as teachers, consider it necessary to employ this particular auxiliary device, but it is of great importance for us to grasp the *spirit* of its usefulness as this is applicable everywhere—viz. that the memory and reason must not be burdened too rapidly with the *simultaneous necessity of mastering a new principle along with the highly contracted symbolism,* in which it is ultimately and conveniently embodied. Preceding the highly finished contraction should often go the more suggestive though clumsier stage of symbolism; provided that there is nothing to unlearn when the final form is adopted, but simply additional contractions introduced, which the learner himself recognizes as reasonable and as a distinct economy of time and space.

Two Extremes in Mathematical Education.

With regard to this essential matter of the appropriateness in *quality* of the material offered for assimilation, it is instructive to note two extremes, equally injurious in educational practice.

The one extreme is the attempt to present a highly *abstract* branch or system of mathematics to pupils of an age not sufficiently mature in brain to assimilate it. This always leads, always has led, to the worst forms of pure rote-work.

As too common examples, I cite the introduction of pupils to a highly symbolical Algebra before they have undergone any preliminary training in Generalized Arithmetic, or, what is often called, Arithmetical Algebra. The result is frequently deep distaste for the subject, disastrous blundering in mere rote-work, lack of grasp of principle, and the feeblest power of applying the symbolism. We have, in fact, all the evils of premature abstraction, than which, to use the telling phrase of Boole, nothing can be more emasculating to the intellect. This defect in the teaching of Algebra has been and to a large extent is so serious that I shall deal with it more fully, in the light both of the historical growth of Algebra and of the history of teaching itself.

The development of Algebra.

As this present essay is not intended in any wise to replace the study of works professedly concerned with the history of mathematics, but rather to stimulate the teacher to their earnest study, I shall not enter into detail.

Referring again to our historical chart (frontispiece), we observe that, among several races in the history of our subject, Particular Arithmetic has developed gradually and continuously into Generalized Arithmetic, passing through the stage of Rhetorical Algebra into a more or less *symbolical* Algebra. By this term Rhetorical Algebra we mean an algebra in which no special symbols and rules other than those of ordinary language and grammar are employed to express general arithmetical truths. An example of Rhetorical Algebra is the statement:—*the square of the sum of two numbers equals the sum of their squares increased by twice their product.* Briefly put (the matter is so important that I have discussed it much more fully elsewhere) the main historical stages may be illustrated thus:—

Particular Arithmetic.	Stage I.	5 cows	+ 3 cows	= 8 cows.[1]
	Stage II.	5 things	+ 3 things	= 8 things.
	Stage III.	5	+ 3	= 8
Generalized Arithmetic.	Stage IV.	$5a$	$+3a$	$= 8a$.
	Stage V.	ma	$\pm na$	$= (m \pm n)a$
		m, n, a, all essentially signless numbers, and $m > n$.		

[1] For fuller analysis of the stages see Appendix, p. 361.

Symbolical Algebra or Algebra proper.	Stage VI. $ma \quad \pm na \quad = (m \pm n)a$, where m, n, a may be any positive or negative numbers whatsoever.

In the passage from V to VI the sign of operation (+ or − &c.) has assumed a twofold function, being used not only as a sign of operation but as *part of the operand itself*.

The radical mistake of algebraical teaching for many generations was in passing by a jump from Particular Arithmetic to purely Symbolical Algebra, and thereby omitting a sufficient training in *Generalized Arithmetic*, without which Stage VI above was often mere meaningless rote-work, for Generalized Arithmetic is the simplest type of a *significant* symbolic algebra. [See De Morgan, *Trigonometry and Double Algebra*.]

These stages might easily be further subdivided; thus arithmetic *without* would precede arithmetic *with* written symbols; and here we get, indeed, a strong hint from history of the value of *Mental Arithmetic*, of which I have already spoken.

Finally, we note, historically, that in all stages of their development Generalized Arithmetic and Algebra have been closely bound up with *Geometry*, theoretical and practical, i. e. applied to measured units of space. The teacher is earnestly advised to study the historic development of mathematics in these, its special aspects, if he desires really valuable guidance in his teaching of these subjects. Happily our best schools have made remarkable progress in the last ten years in the improvement of algebraical teaching, thanks largely to the arguments of Perry and the masterly work of Chrystal and of his great predecessors in this branch of work in England, namely Boole, Peacock, and De Morgan. We now find, in many schools, squared paper for elementary curve-tracing, thus uniting geometry, mensuration, and algebra in one branch of mathematics (viz. Co-ordinate Geometry): but there is still great room for progress even in this direction alone.

The point was introduced here to show how a grasp of the spirit and order of *historical* development of algebra would have enabled teachers to avoid this great blunder in education which has afflicted so grievously whole generations of pupils.

The Teaching of Mechanics.

As another example of the extreme of teaching we are considering, may be briefly mentioned (in addition to the case of Euclidian teaching which we have already considered) the teaching of *Mechanics*, both elementary and advanced. Here the same kind of radical blunder is still frequently perpetrated outside the technical schools—concepts introduced without percepts: abstraction without the underlying concrete basis of experiment. Experience has amply shown that the conventional way of presenting this subject—a way precisely comparable with the introduction of beginners to geometry through 'Euclid'—is ineffective to an extreme degree. The matter has been so forcibly stated by Ernst Mach that I cannot do better than borrow his words :—

'If we know (mechanical) principles like those of the centre of gravity and of areas only in their abstract mathematical form, without having dealt with the *palpable simple (experimental) facts which are at once their application and their source,* we only half comprehend them, and *shall scarcely recognize actual phenomena as examples of the theory.* We are in a position like that of a person who is suddenly placed on a high tower but has not previously travelled in the district round about, and who therefore does not know how to interpret the objects he sees.'[1]

Purely experimental in the initial stages, the teaching of mechanics should throughout, even in the advanced stages (including the Honours Standard offered at the universities), make constant appeal to actual experiment carried on in laboratories, if the effect is to be lasting and a knowledge of Nature, as mechanical, to be something more than words and a highly ingenious knack of solving riders in 'applied mathematics'.

The teaching of mechanics has made considerable progress during recent years by direct appeal to experience and experiment though much remains to be done.

In Chapter XII I have striven briefly to indicate its position in relation to a course of mathematics.

[1] *Mach, The Science of Mechanics* (historically described): this work of genius should be familiar to all teachers of Mathematics and Physical Science.

The Teaching of Practical Geometry.

The other extreme, an *undue* predominance of sense-perception, is well illustrated in those secondary and technical schools and classes where we find a so-called 'Practical Geometry' taught, by pure rule-of-thumb methods, out of all relation to reasonably intelligent grasp of the reason 'why'—i. e. in almost complete dissociation from thought proper. This extreme may also be seen in any badly-organized kindergarten, where the elementary teaching in arithmetic and geometry is so wholly concrete, so wholly based on sense-perception, and such feeble attention is directed to the stimulation of the growth of simple reasoning and imagination, that the child's mind becomes unduly stunted. To put it concretely—the child is forcibly kept playing so long with bricks, bundles of sticks, aimless paper-folding (all of which concrete devices are most excellent aids when appropriately mingled with the stimulating difficulties of abstraction and thought) that at length one finds he cannot think or operate without them. It is a case of arrest of development—the shaded band in his developmental chart has not increased proportionately to the unshaded—he becomes the adult who *can count only on his fingers*. In terms of racial development, we may fitly describe such children and adults as, in this respect, *primitive* folk.

The *practical* effect of either extreme of education is —incapacity to *apply* the knowledge to new experience. Rationalized skill and realizable thought—the harmonious combination of *doer and thinker*—that, I take it, is our aim in mathematical education.

Subsidiary Principles.

A few corollaries may suitably be drawn here from our central principles. Here again I would repeat in substance what I have already remarked: that, though personally I consider the central principles in themselves sufficiently firmly established by experience as valid principles for educational science, and therefore any logical deductions from them alike permissible and valuable for practice, with such common-sense modifications as will be applied by an experienced teacher, yet it is not necessary, nor perhaps even advisable, that the teacher should, just yet, so regard them.

Let him rather view these subsidiary principles as themselves standing firmly on their own feet, as having been successfully applied for many years by efficient teachers; principles, the *value and truth of which the teacher may, by actual trial, test for himself*: thus only, indeed, will he come to have faith in them. Nor let him be dismayed if many of them appear contrary to his own past experience, not only at first sight, but perhaps even after some little practical trial of them. Let him persevere. For I am convinced from lengthy experience with pupils of all ages and many kinds of capacities that his perseverance with the trial will, in the end, amply reward him, both in the increased pleasure he will derive from his teaching and the increased interest, joy, and capacity he will observe among his pupils.

Ist Principle. *From the very beginnings* (in the Kindergarten) *of, and throughout mathematical education, the pupil should be guided to the* DISCOVERY *of definitions, theorems or truths, rules of operation, &c., for himself, by a process that has historical warrant—viz. induction from a number (large or small, as the case may necessitate) of favourable particular cases gradually increasing in complexity.* Excellent historical examples are provided by the Development of the Pythagorean Theorem, the Index Notation, the Exponential Series, the Binomial Theorem, D'Alembert's Principle in Theoretical Mechanics, and so on.

Historically and logically, definition is simply a refined species of *description*, a process, in the evolution of knowledge, of increasing definiteness, in enunciation of statement, *which never ceases.*

The effective application of this principle would at once render superfluous half the examples now found necessary in the way of drill in the application of principles and methods. The other half would, of course, still be found necessary as drill as all experienced teachers know.

As teachers, too, we aim throughout at making the pupil more and more independent of ourselves.

If this principle is to be successfully applied, the teacher must beware of aiming at a rigour of definition and of proof beyond the capacity of his pupil to assimilate. Indeed, the real tests of assimilation are (i) capacity in the pupil to discover (under the teacher's guidance) the truth for himself, and (ii) capacity to apply it to new experience; and, in this

sense, the principle contains its safeguard partly in itself. Still, much valuable time and energy on the part both of teacher and taught may be saved if the teacher remember that there is a *degree of refinement of definition and of rigour of proof appropriate to each particular age of the individual.*

Historically, this is one of the best-established facts :—broadly speaking, that from the top of our central historical chart onwards to the bottom, rude description gradually passes into refined definition, and crude experimental evidence or proof through intuitional insight on to systematic and scientific rigour: but the process of increasing subtlety of definition and rigour of proof still continues and has, in its very nature, no end. Each generation, while refining on its predecessors, vainly imagines the last link of proof has been forged by itself, but the succeeding generation invariably advances, or should advance, one step further—in fact the continuity of the process is the test of progress. Once he grasps the fact that perfect definition and rigorous proof, being ideals, are never attained or attainable, the teacher at length recognizes the educational corollary derivable therefrom—*fit the proof to the brain maturity and experience of the pupil.*

Not only has the old school teacher, with his prematurely presented Euclid and his symbolical algebra, neglected this fundamental condition for efficient assimilation of knowledge, but the same blunder is committed by the college professor who vainly attempts to instil into the student refined subtleties which he, the professor himself, perhaps only half understands, even after years of specialist study. The same educational blunder—for it is in the school for the adult specialist that the genius is master and teacher—produced the unhappy fate of such a genius as Grassmann, whose rigour of proofs and generality of idea were entirely beyond the capacity of his contemporaries to assimilate, so that his truly sublime writings in mathematics had at least a generation to wait before they began to be assimilated.

I repeat, finally, that the somewhat natural tendency in teachers of every grade of mathematical study to *insist always on that essence and form of proof which, in the teacher's opinion, is the most logically rigorous, is a constant, deep-lying, and unsuspected cause of perhaps the greater part of the strain, ineffectiveness, and monotony of mathematical education.*

II[nd] Principle. *No symbol or contraction should be intro-*

duced till the pupil himself so deeply feels the need for such that he is either ready himself to suggest some contraction, or at least appreciates reasonably fully the advantage of it when it is supplied by the teacher.

Even the *laziness* of the young may be here utilized, for the appropriately chosen symbolic contractions save labour—a function which, for the adult who employs them as such, we dignify by the title of *economy of thought.*

Even when the new symbol *is* supplied, the older and more cumbrous form of expression should be employed concurrently with the new, till familiarity with the new and its superiority in economizing labour or space or both drive the older mode of expression naturally from the field.

Familiarity with the principle symbolized, in general, historically precedes its conventional symbolization. Thus the grasp of the fact that the order of multiplication is indifferent long preceded its symbolical formulation.

$$a \times b = b \times a.$$

Even Newton, the inventor of the *general* exponent (a^n or a^x) uses $aaaa$ for a^4 *concurrently* with the form a^4. This principle has deep historical foundation, and, wisely employed, lessens enormously the difficulties of mastering new rules of operation consequent upon the introduction of new symbols.

EUROPEAN EVOLUTION OF THE 'INDEX' IN ALGEBRA.[1]

Stifel & Recorde. About 1550.

$$\left\{\begin{array}{llllll}
\mathfrak{c} & -\,2\,\mathfrak{z} & +\,3\,\not{\mathfrak{z}} & -\,4\,\varphi & & \\
\mathfrak{z} & +\,\not{\mathfrak{z}} & +\,\varphi & & & \\
\hline
s\mathfrak{z} & -\,2\,\mathfrak{z}\mathfrak{z} & +\,3\,\mathfrak{c} & -\,4\,\mathfrak{z} & & \\
 & +\,\mathfrak{z}\mathfrak{z} & -\,2\,\mathfrak{c} & +\,3\,\mathfrak{z} & -\,4\,\not{\mathfrak{z}} & \\
 & & +\,\mathfrak{c} & -\,2\,\mathfrak{z} & +\,3\,\not{\mathfrak{z}} & -\,4\,\varphi \\
\hline
s\mathfrak{z} & -\,\mathfrak{z}\mathfrak{z} & +\,2\,\mathfrak{c} & -\,3\,\mathfrak{z} & -\,\not{\mathfrak{z}} & -\,4\,\varphi \\
\hline
\end{array}\right.$$

φ = 'absolute'
$\not{\mathfrak{z}}$ = the root
$\mathfrak{z}$ = square (Zensus or census, x^2)
$\mathfrak{c}$ = cube
$\mathfrak{z}\mathfrak{z}$ = zenzizenzike
$s\mathfrak{z}$ = the sursolide

[1] After Heppel [*Nature:* May, 1893].

Stevinus. 1586.

$$\left\{\begin{array}{l}
\textcircled{3} - 2\,\textcircled{2} + 3\,\textcircled{1} - 4\,\textcircled{0} \\
\textcircled{2} + \textcircled{1} + 1\,\textcircled{0} \\
\hline
\textcircled{5} - 2\,\textcircled{4} + 3\,\textcircled{3} - 4\,\textcircled{2} \\
\qquad + \textcircled{4} - 2\,\textcircled{3} + 3\,\textcircled{2} - 4\,\textcircled{1} \\
\qquad\qquad + \textcircled{3} - 2\,\textcircled{2} + 3\,\textcircled{1} - 4\,\textcircled{0} \\
\hline
\textcircled{5} - \textcircled{4} + 2\,\textcircled{3} - 3\,\textcircled{2} - \textcircled{1} - 4\,\textcircled{0}
\end{array}\right.$$

(After Viète, 1591.) Harriot. 1631. In use down to about 1750.

$$\left\{\begin{array}{l}
aaa - 2aa + 3a - 4 \\
aa + a + 1 \\
\hline
aaaaa - 2aaaa + 3aaa - 4aa \\
\qquad + aaaa - 2aaa + 3aa - 4a \\
\qquad\qquad + aaa - 2aa + 3a - 4 \\
\hline
aaaaa - aaaa + 2aaa - 3aa - a - 4 \\
\hline
\end{array}\right.$$

Descartes. 1637.

$$\left\{\begin{array}{l}
a^3 - 2a^2 + 3a - 4 \\
a^2 + a + 1 \\
\hline
a^5 - 2a^4 + 3a^3 - 4a^2 \\
\qquad + a^4 - 2a^3 + 3a^2 - 4a \\
\qquad\qquad + a^3 - 2a^2 + 3a - 4 \\
\hline
a^5 - a^4 + 2a^3 - 3a^2 - a - 4
\end{array}\right.$$

Newton, 1642–1728, invented finally the general exponent (as an arbitrary constant or as a variable) as :—a^n, a^x; the latter the so-called 'exponential' function.

FIG. 90.

The above diagram, which gives four kinds of successively developed symbolisms for writing one and the same algebraic content, is, with a little careful study, self-explanatory.

Note the now apparently simple but, at the time and in its results, deeply important invention by which *letters of the alphabet came to stand for unknown quantities* (employed by Viète, the great French mathematician, and subsequently introduced into England by Harriot). It is very characteristic of mathematical history, where advances and inventions are repeatedly lost and re-discovered or re-made, that the still greater invention by which letters of the alphabet were used for *known as well as unknown* quantities had been *previously* made by the great German mathematician Regiomontanus (1534). The teacher should note also, that many species of symbolism were from time to time introduced which, from unsuitability, prematureness, or for other reasons, were not destined to survive, or even to fulfil the almost as important office of suggesting improved developments leading to the ultimately accepted form. When I say 'ultimately', I do not intend the word to be interpreted as absolutely final, for we have no sound reason for believing that the present forms a^2, a^3, a^4 . . . are destined to survive perpetually. Indeed the whole history of symbolism rather indicates the truth of the opposite view, viz. that every mode of symbolism is, in the long run, transitory, and becomes gradually replaced by superior modes. This feeling, impressed upon the pupil, begets a liveliness of interest and sympathy with the science as the continual growth of a *living* thing, instead of the generally felt monotony incidental to a supposed perfected and dead mechanism.

I append a few additional historical symbolisms :—
Bombelli (Italian) writes, 1572,

$$1\ \underset{\smile}{2}\ \text{p.}\ 5\ \underset{\smile}{1}\ \text{m.}\ 4$$

for the modern x^2+5x-4 (using p. for plus, m. for minus). A little previous to this, the same expression would have been written :

$$1Z\ \text{p.}\ 5R\ \text{m.}\ 4$$

(where Z = zensus or census or x^2, and R = radix or res or x). Sometimes for Z stood C (census) and for R stood Rj. Viète himself denoted the different powers of A by A, Aq (q = quadratus), Ac (c = cubus), Aqq, and so on, which usage would suggest the improvement noted

in the above Table and due to Harriot (the Englishman). Descartes' improvement (a^2, a^3 . . .) combines the fundamental idea of Bombelli and Stevinus (a number to indicate the power) with that of Viète and Harriot (the clear indication of the letter whose power we are dealing with). (See also Appendix for fuller historical illustration.)

My own experience, as regards this second subsidiary principle, is that the teacher will find it highly useful, and, in the long run, *economical of time*, to allow his beginners in Generalized Arithmetic to use the longer but more suggestive symbolism *aa*, *aaa*, *aaaa* . . . , until they have multiplied and divided with these powers so frequently that the rules of operation $a^m \times a^n = a^{m+n}$, $a^m / a^n = a^{m-n}$ (if $m > n$) $= 1 / a^{n-m}$ (if $n > m$), $(a^m)^n = a^{m \times n}$ and so on, are *without difficulty discoverable by the pupils themselves* directly the contracted notation a^2, a^3, a^4 . . . is introduced. Thereafter practice will make still more perfect, and one does not then meet with those blunders (like $a^3 \times a^3 = a^9$, and so on) incidental, almost inevitable, to the usually hurried and often dogmatic way of presenting these valuable symbols.

It should be both present constantly to the teacher's mind and often impressed upon his young pupils that mathematical symbols have largely been created, and are still constantly being created, to relieve the mechanical labour of writing and thinking—*to economize space and time and energy* (this, of course, is true not only of algebra but of geometry). But this fundamental advantage is largely lost for those who have not intelligently mastered the meaning of the symbol and gained reasonable accuracy and dexterity in the use of it.

I have frequently had excellent symbolic contractions invented by lads sharp enough to utilize their heads to save their fingers, and not rarely the symbol invented has been one of those in use, perhaps centuries ago, by professional mathematicians themselves.

The above considerations also lead to the following :—

IIIrd Principle. *Only that extent of syllabus should be attempted which the pupil can* (i) *rationally grasp the basis of*, (ii) *attain mechanical dexterity in the use of, for subsequent application—each to at least such an extent that his education therein may consciously stand to him as a genuine type of what practical and effective mastery of a subject really implies.*

This practically implies varieties of syllabuses corresponding to the respective grades of general education and technical training (male and female). And it may be added that specific education has missed its chief aim if there does not remain *at the end both the desire to continue its development and the capacity to do so.*

The first part of this third principle is closely related to the first principle, and particularly to the important truth as to the *relativity of proof* thereto appended. I leave the obvious application to the reader.

The two extremes here are, on the one hand, mechanical dexterity with little or no power to apply it, through lack of rational grip of its basis and significance; and, on the other hand, mere unintelligent grasp of principle without practical mastery of it as a machine for application. Educational history affords numerous instances of each extreme.

Observe that all these principles, if applied, tend to the development of sturdy inventiveness and self-reliance in the pupil.

IV^th Principle. *Make the pupils gradually familiar with the great names, the interesting facts in mathematical history—the long struggles, the frequent blunders, the final successes of his ancestors in building up the science. Let him note the value of international and inter-racial co-operation and emulation in this grand task.* Thus will his sympathy be aroused: the subject becomes living and inspires strenuous effort. With this principle is also closely allied the *aesthetic* appreciation of mathematics, which is capable of considerable development even in the very young. On this I have dwelt elsewhere.

CHAPTER XVIII

FURTHER EDUCATIONAL APPLICATIONS

The Correlation of Studies.

By *correlation* I imply the existence of (i) a *direct* bond of connexion between the different branches of so-called pure mathematics one with another (e.g. of arithmetic with algebra, and of both with geometry in mensuration, and so on), and (ii) equally important, but most neglected, a *direct* bond between the branches of pure mathematics themselves on the one hand, and the branches of applied mathematics and other closely related studies on the other, e.g. Practical Geometrical Drawing (plane and solid), Physical Geography and Astronomy, Surveying, Mechanics (theoretical and experimental), Physics, Art, Wood and Metal and other Manual Work, Statistics (social and other), &c. There is also the further problem of the bond between mathematical education and all other studies, and finally with life outside the school or college. This is, of course, indirectly implied in the present investigations, but it is proposed to deal with it more fully and directly in a work the writer is preparing on education in general.

Looking at many institutions, one might be tempted to believe that their organizers considered these subjects had been really ' correlated ' by simply allotting hours to each in the curriculum. We may be sure, from sad experience, that unless the teachers themselves see that the correlation is genuine—and not merely of this time-table kind!—the pupils will never create such connexions in their own mind as will round their mathematical studies into an intelligible systematic unity of experience.

Stimulate the free interchange of ideas between the pupil's varied departments of school studies themselves, and between these and his out-of-school life, remembering that though it is necessary to isolate school studies from each other and from

the world in order to conquer in detail, it is equally necessary to be continually reuniting the parts, lest we finally fashion a being whose intellect is as a house with many chambers lacking doors and windows alike.

What are the Tests of True Correlation?

Our historical parallel gives us these at once. For, when we affirm historically that surveying and geometry were correlated in Archaean civilization, we mean that practical problems raised in the sphere of surveying necessitated for their solution the existence of appropriate geometrical devices; and, further, that such geometrical devices, if not already evolved by other agencies, were created for this special purpose. In the latter case surveying reacted upon geometry itself, in adding to its existing stock of symbols, ideas, methods and results.

We may therefore formulate the fundamental test of true correlation thus :—

Wherever a problem, raised in one branch of experience, necessitates, in some other branch, the existence, or the creation if non-existent, of a machinery of ideas, symbols, methods, and results, appropriate for its solution, there we have true correlation between these two branches of knowledge.

For our present purpose, we take the second named branch of experience as pure mathematics. Two distinct cases then arise :—

(i) Where the appropriate machinery already exists, we have simply an application of mathematics. There the matter ends so far as mathematics is concerned.

(ii) But where the appropriate machinery has to be created, *mathematics itself receives a new development.*

From the educational standpoint of the individual, in the first case the pupil is practised in the application of already acquired mathematical knowledge; in the second he is confronted with a novel and interesting problem whose solution compels a further development of his own mathematical knowledge. Both cases of correlation are obviously indispensable to education, but the second is much the more valuable owing to its greater stimulus to originality, self-reliance, interest, and genuine mastery.

With respect to the most effective means for the development of mathematics, I venture on historical, psychological,

and experiential grounds, to formulate the ideal of mathematical education thus :—

All branches of pure mathematics, from the infant's experience of simple counting in the nursery to the student's study of the highest branches of analysis at college, should be slowly and continuously developed by the pupil himself—under the wise help and guidance of parent and teacher—as the appropriate instruments for the solution of problems presented in the individual's attempts to understand and interpret his social and physical environment.

The gradual organization of plans, suitable to all grades and varieties of education, on the lines above indicated, is a task that, in germ, is already being successfully developed in a few schools and colleges, though the aim has not always been distinctly formulated. Its complete solution is, of course, an ideal, for new conditions (as the rise of new occupations and other far-reaching causes) will necessitate ever new modifications. Nevertheless, I believe that the next generation of teachers, by conscientious and continued efforts, with sufficient stimulus from the public, and liberty of development from the educational authorities, will succeed in realizing a considerable part of the programme thus generally outlined.

Some Illustrations of Correlation.

Correlation between simple Descriptive Geometry, Drawing, and the Physical Environment.

(*a*) In the Bulgarian Pavilion of the Paris Exhibition of 1900, I noted what appears to me to be a true and useful correlation. The problem is to draw a window (see Fig. 91). The pupil is asked, first, to *describe* the shape of the window as it exists. The attempt to do this demands the analysis of the whole into geometrical parts (points, lines, surfaces, &c.). Then comes the distinction between curved and straight lines, presumably obtained from the pupils themselves by appropriate questioning. In this way, *vague description passes gradually into comparatively clear definition.* The geometrical ideas thus gained are then applied to more and more complicated

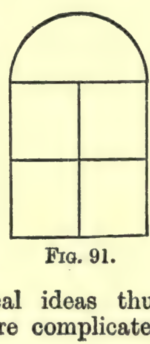


FIG. 91.

cases (e.g. jugs with handles, an easel, rows of palisades, a simple cottage, &c.), necessitating the development of still more complex geometrical conceptions (solids, &c.). Here we have an excellent inter-play between sense-perception and intellectual conception.

(*b*) In experimentally discovering and establishing the truth of Boyle's Law in physics, pupils plot the results on squared paper. The investigation of the resulting curve would lead to a new idea (assuming it has not already been introduced in mensuration problems): the conjunction of algebraical equations with geometrical curves—the basal-idea of analytical geometry. Here, then, would be true correlation between algebra and geometry, and also between these, united, and physics.

Geometry (theoretical and practical) and Geography.

(*c*) A fundamental problem in geography is the determination of the relative positions of myself and of natural objects on the earth's surface. *Where am I? What is my position, here and now?* Place any simple object on the school-room floor, and ask one of the children to describe its position so that, if taken away, the object could be placed precisely as before.

After some discussion to clear up the exact nature of the problem, you will finally elicit from the children these facts:

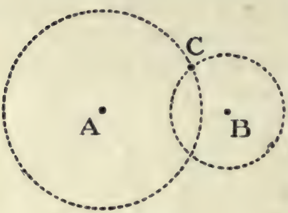


Fig. 92.

1. To describe the position of one object (C, say) on the floor, one must already know the positions of two other objects on the floor (A, B, say).

2. Two measurements must be taken, of which one at least must be a line-measurement, or distance.

3. If one measurement only (say, distance between C and A) is known, then, by *experiment*, the child finds that C may lie anywhere on a certain curve which is recognized as identical with a previous sense-impression, viz. the boundary of the plane section of a ball or sphere, in fine, a *circle*. This at once leads to fuller descriptions of the properties of a circle (as a locus). The description, by

contrasting it with another object liable to be confounded with it in certain aspects (a sphere), passes finally into a passably good *definition*.

4. If we know two measurements, viz. AC and BC (and have, of course, a qualitative statement to distinguish the two intersections), then the child describes the two circles and the problem is solved.

A similar discussion with another frame-work of reference (viz. a pair of meeting or intersecting straight lines, the well-known Cartesian axes, for which the intersections of two walls with the floor may stand) evolves simultaneously the ideas of parallel lines, parallelograms, and a *second* solution of the initial geographical problem.

Finally, the substitution of a spherical blackboard for the floor or plane blackboard, aids in evolving the conventional solution of the problem for the earth, by lines of latitude and longitude. I have already dwelt in some detail on the *fundamental place that the geometry of the Determination of Shape, Size, and Position should occupy in all, and especially the early stages of mathematical education.* (See Chaps. XI, XII.)

False Correlation.

It is perhaps now scarcely necessary to point out any test of falseness of correlation, seeing that such is implicitly contained in the above-stated test of the true. Nevertheless, falseness of correlation so abounds that I touch upon it a little. As a rule, it is a *useless residue of what at one time was a genuinely useful correlation*—a functionless survival, like the human tail. For instance, in the days of the Hanseatic League, arithmetic was often taught to youths in mercantile offices explicitly for mercantile purposes, so that the examples and applications were not only familiar to them, but had direct bearing upon the future occupation of the student. These examples and applications still persist largely in our arithmetic books and lessons, in the shape of curious tables, mercantile problems, and the like. But it is seldom seen that the justification of their existence has largely vanished: as is so often the case, the letter remains but the spirit has gone. Other instances of this useless survival are—the G.C.M. algorithm, square and cube roots (by algorithm), vulgar fractions with large denominators

(necessary before the invention of decimals), clock-problems (necessary in the days of clepsydras), Compound Proportion, the calculation of Compound Interest without logarithms, the dot-method in Rule of Three (necessary when proportion was dependent on Euclid's theory of ratio), &c.

Amusing instances of this false correlation, where clearly a special effort has been made by the teachers to make the correlation conscious, were to be seen by the inquiring visitor at the Educational Exhibit of the Paris International Exhibition of 1900. I noted down several at the time, and append one.

The aim (as stated by the teacher very plainly in a short introduction to the pupils' notebooks on exhibition) was to correlate geography and arithmetic.

TEACHER: Where does wool come from?

PUPIL: Sheep.

TEACHER: What do you call the man who cuts the wool?

PUPIL: The shearer.

TEACHER: If a piece of woollen cloth 288 yards long be divided among nine people, how much does each get?

[Answer worked out as 32 yards.]

This kind of correlation is so only in name: it is false and artificial, and injurious as diverting attention from the true kind.

Broad Solution of the Correlation Problem.

For a broad solution of the very difficult problem of genuine correlation, our central historical chart is highly suggestive.

(a) *Correlation of the Branches of Pure Mathematics amongst themselves.*

I have dealt with this already. Here suffice it to say that the injurious degree of isolation of arithmetic, algebra, geometry, trigonometry, &c., that at one time existed is rapidly being broken down, and we are tending to a unification of all branches into one *whole* of mathematical science and art. The key-note of the reform may be justly described as the *Arithmetization of Geometry*, and the *Geometrization of Arithmetic*, in mensuration, curve-tracing, &c., by the intermediacy of Generalized Arith-

metic and of Algebra. This modern reform movement has full warrant, in the light of our fundamental historical parallel-principle, from the teaching of history. But it must never be forgotten that this unification is not only consistent with but actually implies the isolation, from time to time, of each branch for concentration and mastery.

(*b*) *Correlation of Pure Mathematics with Applied Mathematics, or with Experience demanding Mathematics.*

As a broad solution to this—much the more difficult problem of the two, and depending for its detailed solution largely on success in the present reform movement just briefly mentioned—I would venture to propose the following :—

Group together (on the central historical chart) *the right-hand line of individual development with the left-hand line of external factors that have actually been effective in developing mathematics in the race—remembering that a factor once started is to be continued.*

The resultant grouping will broadly run thus :—

Group I (*for children of both sexes*).

The Mathematics of Infancy and Childhood in correlation with Primary and Derivative Occupations, and the very elements of Geodesy.

Group II (*for boys*).

The Mathematics of school in correlation with the preceding developed to a higher stage along with Mechanics, to which latter predominance is to be allotted—the term 'Mechanics' covering experimental and theoretical Kinematics and Dynamics, including the elements of Physics and Astronomy, the Mechanics of the heavens.

Group III (*for young men*).

The Mathematics of college in correlation with the preceding developed to a still higher stage, along with Physics, to which latter predominance is to be allotted.

The above groupings are concerned with general education only. The problem of technical groupings is not specifically

treated, being easier in general, and subject to rapid experimental solution.

Types of Mathematical Education for Male and for Female.

The writer has had considerable experience in the teaching of girls and young women, but he does not feel competent to offer any but the broadest suggestions upon the problems concerned with their mathematical education. That the infancy and early childhood of the two sexes may appropriately share their education under the guidance of the woman is now a just verdict of experience, I believe. But the questions of the development of the appropriate grouping and types of mathematics for the female sex when these early stages have been passed are in a condition still more obscure and immature than the corresponding questions for the male. This is not to be wondered at if we look back upon the history of education. Doubtless the solution of these problems for women can only come, and gradually, from women themselves, when more emancipated from their present-day adherence to and imitation of male conventions and standards in education.[1] Considerable research work in the history of mathematics, particularly in relation to the rise and growth of female arts and crafts, deliberate experiment with carefully-planned varieties of curricula, and woman's confidence in her own power to work out the best solutions, are undoubtedly all needed here. Of course there are certain fundamental principles that will be found to dominate all types of mathematical training—male and female, general and technical, some of the chief of which I have striven to indicate. But within these broader principles subordinate but highly important special truths and applications will doubtless emerge, differentiating mathematical education for boys and young men from that for girls and young women, and within these broad groups again, further differentiating the mathematics and methods of teaching mathematics for general culture from the corresponding demands for technical training. Thus the large problems in this whole matter are systematizable in the following table :—

[1] There are many signs that this emancipation is being realized, now that women have shown their capacity to master the highest academic studies.

<table>
<tr><td colspan="3">Male.
↓</td><td colspan="3">Female.
↓</td></tr>
<tr><td colspan="6" align="center">Infancy and Childhood.</td></tr>
<tr><td>Boys:</td><td>General</td><td>primary.
lower secondary.
higher secondary.</td><td>Girls:</td><td>General</td><td>primary.
lower secondary.
higher secondary.</td></tr>
<tr><td colspan="3">↓</td><td colspan="3">↓</td></tr>
<tr><td>Youths (male):</td><td>Technical (including professional)</td><td>primary
secondary
tertiary.</td><td>Youths (female):</td><td>Technical (including professional)</td><td>primary
secondary
tertiary.</td></tr>
</table>

Each side involves six broad problems; thus, with that of infancy and childhood, there are thirteen problems here awaiting appropriate solution. Doubtless each will illuminate the others. Particularly is light to be expected from the discovery of the mutual relations between the general and the corresponding technical standard, namely the relation of primary general to the subsequent primary technical, of lower secondary general to the subsequent secondary technical, and of higher secondary general to the subsequent tertiary technical.

These two groups will naturally act and react upon each other continually to the ultimate benefit of each, both on the male and on the female side.

Remarks on the special significance of the Groupings above.

It is *not* suggested that every pupil shall pass through each of the three groups I, II, III: that is clearly an absurdity. It will always be a comparatively small number, for instance, who need the last group. The element of *Occupations* running, *in essence*, right through, is to be interpreted in Groups II and III as indicating the need for the close correlation of mathematical teaching with the future demands of the occupation, assuming that the pupil has the chance given him of early showing aptitude for some particular occupation, and that this occupation demands a certain mastery of certain branches of mathematics. The stress to be laid on this element of occupation will, of course, vary according to the age at which the lad is likely to leave school, and the position his abilities or social rank or both destine him to fill.

For the masses—especially those who will become skilled artisans or workers—great weight should be laid on the close correlation of mathematics with the prospective occupation during the last two years or so of school life, the actual subjects correlated being some form of *manual* work and the elements of *Art*, along with (as already stated) experimental and theoretical mechanics for boys. In the school of the near future, creative manual work with Mathematics as its servant and joyful Art as its mistress will occupy a large place in the various school curricula of the masses.

For those who reach Group III, the weight on occupation will fall in this last stage. For instance, in the training of the future engineer there should clearly be the closest co-operation between the mathematical studies and the demands of engineering science and practice, along with predominant stress on physics (including the higher branches of mechanics). In the training of the future teacher of mathematics as a speciality, again the occupation should be weightily considered; in this case still greater stress should be laid on mechanics and physics (experimental and theoretical, of course).

Returning to Group II, I believe that the experience of science-teachers, apart entirely from any historical parallelism, warrants us (as meeting present-day requirements) in summing up the aim here by suggesting, as an experiment, such an organization of the school-staff *that, for boys, the teacher or teachers of mechanics with properly equipped laboratories and drawing rooms be also the teacher or teachers of mathematics*; or, preferably, perhaps, that the two kinds of teachers, after consultation with the manual, art and science masters, jointly produce a syllabus of mathematics and mechanics.[1]

Finally, I offer a few remarks on the interpretation of Group I.

Group I.

The Young Child and his Environment: a model, in

[1] Already realized in some schools. The problems involved in uniting the co-ordination principle with the broad groups of talents of the pupils also, however, require thorough consideration and experiment. Thus the proper mathematical education of the artistic pupil is a pressing and difficult problem.

essentials of correlation, &c., for further development in School and College, and in after-life.

Just as the struggle of primitive man with natural difficulties stimulated the growth of mathematical knowledge, so is it with infancy and early childhood. I have already had occasion to remark upon the wonderfully rich mathematical experience exhibited by quite young children, all of which is entirely evolved in their serious play and struggles with their physical, and, in a less degree, their social environment in the home—the child's world.

This process is *interesting*, *rapid*, and *thorough* : it results in *practical mastery* of their mathematical knowledge, and in a high degree of *development of inventiveness, self-reliance, and assurance in acquiring it.*

In all these respects is it not a model for subsequent development in school ? Cannot the school curriculum be so organized as to admit the play of factors essentially similar to those that underlie such rapid and thorough mastery by the young child ?

Of course, while developing originality, independence and confidence in our pupils it is advisable to remember that virtues to be effective should be developed in complementary pairs, difficult as this unfortunately is. There is therefore equal necessity for the development of the power of imitation, desire for co-operation, and genuine humility. Indeed I venture to assert that the first group will rarely reach the highest possible degree of development unless accompanied by an equally vigorous growth of the second.

What does this imply ? Surely—to the thoughtful teacher—an organization of mathematical education as far as possible in harmony with the principles I have been throughout dealing with. Moreover, with the addition of more systematic guidance : with the teacher partly co-operating with and partly replacing the parent : with the rapidly increasing ratio of conceptual to perceptual experience—in no wise inconsistent with the rapid increase in each element—the *rapidity* of development in kindergarten and school should far exceed that in the nursery.

Returning now to the correlation of the mathematics of childhood with primary and derivative occupations, and simple geodesy, we cite as details the child's use in nursery and kindergarten of beads, tallies, bundles of

bound sticks, the abacus, simple strokes or dots for units (//// . . . .)—all used by primitive races—paper-folding, mensuration of actual concrete objects (floors, walls of schoolroom), simple surveying (height of school-house, distance between inaccessible objects) by use of angle and cord, the geometry involved in fundamental problems of geography (those relating to the fixing of position—the elements of geodesy), experimental determination of areas and volumes, description and measurement of physical forms, simple modelling and drawing and the geometry involved therein—all such we cite as evidence of a process of development strictly analogous to the development, in the early history of man, of mathematical experience through the stimulus arising from his occupations; for '*the child is father of the man*' and '*the worker is the father of all men*'. Such a training is for hand and eye, and thought, alike: sense-perception and conception, the concrete and the abstract. At no stage of mathematical education can either factor be with impunity neglected, though the proportions in which they mingle and enter into the experience will rightly vary at different ages. This brings us back to the use of the historical chart as a suggestive help to the discovery of the appropriate proportions of hand and eye exercise (sense-perception) to thought-exercise (conception) at any given age.

In conclusion, if any teacher distrusts the stability of the structure I have erected on our central principle of parallel between race and individual, I would at all events earnestly urge him to a practical trial and test of the principle in a modified form. Let him make a list of those points in mathematical education where special difficulty is ever found by his pupils (e. g. perhaps zero and place-value; long division; fractions; negative, fractional and imaginary units; indices; decimals; logical deduction; generalizations; infinitesimals, limits, functions; &c.). Let him now consult a friend who is *really* familiar with the spirit and development of mathematical history, and ask him to jot down a list of the discoveries which were the most difficult to make and to popularize when made. Compare the two lists: a very striking resemblance will assuredly be found. Let him now employ with his pupils the spirit of the devices that overcame the difficulties in the historical development. Thereafter, I venture to think, he will be a staunch upholder of the value of the principle.

CHAPTER XIX

CULTURE AND OCCUPATION

I HAVE repeatedly pointed out the fertilizing effect of the activities of the various occupations on the growth of mathematics throughout all its branches, from the elements of counting to the higher analysis. This new mathematical thought thus created gradually passes from its proximately original source in some particular occupational activity into the general framework of mathematical science wherein it is finally accorded, often after substantial transformation, a definite position. Much of it thus ultimately finds its way, through schools and other socializing institutions, into the stock of general human culture.

This whole process of socialization of mathematical science is partly automatic and mechanical, partly the result of systematic and conscious endeavour. The growth of civilization is the more rapid, the greater the share of this latter factor. This general process is of immense importance to mankind, and appears to be characteristic of other branches of culture. It therefore seems advisable to throw the matter as briefly as may be into clearer prominence. There are, it would seem, three main stages :—

1. The practical demands of some particular occupation stimulate its followers to the creation of some new branch of knowledge. Let us call this 'utilitarian' knowledge, or A.

2. Some, if not all, of this new thought, A, is, often after substantial modification, co-ordinated with and becomes a part of known science in the hands of professional scientists (who may or may not be experts in that occupation). This let us call 'scientific' knowledge, or B.

3. Some of B gradually (also often after substantial transformation) wins an authoritative place in the schools or other educational agencies, and thus finally comes to form part of what we know as general human culture. Let us denote it by C.

The process is then repeated on a proportionately higher level of knowledge, and therefore of skill.

Suppose the occupation in stage 1 is that of a shipbuilder. Then A, by passing through the stage B and becoming evolved into the final form C, has lost all direct and apparent connexion with the art of shipbuilding. So far as mathematics is concerned, C may be considered the basal element contributed to mathematical culture by the art or occupation of shipbuilding. C having become part of general culture, the future generations of shipbuilders start with a greater command of mathematics, and so A continually tends to increase. Thus the cycle repeats itself. This is merely an illustration. Other occupations likewise contribute their basal culture elements to civilization.

Clearly the type of mathematics denoted by A is technical or occupational mathematics: the type denoted by C is part of the general educational substance. The intermediate grade B, being professional, is, strictly speaking, technical or occupational also, but clearly of a special type. It may be termed semi-technical.

Here we may gather further insight into the difficult problems of general education in its various grades, and, in contrast therewith, of technical training for the various groups of occupations. The more deliberate and systematic the technical training for any particular occupation becomes, the more rapid is the growth of the basal elements contributed by that occupation to the general stock of human culture, and, reciprocally, the more socialized, or, let us say, civilized, do those pursuing that occupation become, and the more effective their activities.

On the other hand, neglect of specifically technical training tends to desocialize proportionately the occupation; and science, humanity, and the occupation itself, all equally suffer from the loss.

Here again we see the profundity of the mutual relation between the progressive evolution (or, it may unfortunately be, decadence) of occupational skill, of science, and of general civilizing culture.

A great future undoubtedly awaits research into the relations between these regions of human activity.[1]

[1] See the writer's *Janus and Vesta*, Chap. XII (Chatto and Windus).

CHAPTER XX

ON THE NATURE OF GEOMETRICAL KNOWLEDGE, AND OF ITS PROCESS OF DEVELOPMENT IN THE INDIVIDUAL[1]

1. *An essentially Important Question: What is the Nature of Geometrical Knowledge?*

If our aim is to grasp the order and form in which geometry can be presented most efficiently for educational ends, it is, I presume, obvious that we must, as teachers, have a working conception of the *nature* of that province of knowledge with which we are concerned—its nature as determined by logic and psychology. Whether we look back upon the misconceptions prevalent in the history of mathematics itself, or the many strange, not to say monstrous, perversions of its educational functions, we are tempted to believe that in no branch of study has so much misapprehension existed as to its nature as in this of mathematics.

2. *What is the Nature of the Process by which Geometry is Developed?*

The outcome of this is, naturally, an equally great misapprehension of the nature of the *mental processes by which alone geometrical knowledge can be genuinely assimilated.* This, of course, reacts injuriously on practical education.

It is these two questions I now propose to discuss.

In the use—here inevitable—of technical philosophic terms, there is perhaps but one way of avoiding a fatal misunderstanding at the very outset—the copious use of illustrations. Preliminary *definitions* are here quite beside the mark. For in no science, not even mathematics, does the definition of a term carry with it its full content for all

[1] Reprinted with modifications by kind permission of the Editor from *The Journal of Education.*

the uses to which it may be put. To such a pre-eminent degree is this the case in philosophy—our present concern—that, as Kant remarks in his *Critique of Pure Reason*, 'a full and clear definition ought, in philosophy, rather to form the conclusion than the commencement of our labours.'[1] The practical outcome of this truth is that we are forced to depend upon the comparison of contexts for grasping the precise significance in which an author uses a technical term —a process essentially identical with that undergone in learning a language. Now, the comparison of contexts is, in effect, a cumbrous and tedious form of the use of direct *illustrations*. So, without further apologies, I begin with a copious use of illustrations.

3. *Illustrations of the Philosophical Terms to be used.*

My object is to select certain pieces of geometrical knowledge, or, let us say, spacial experience, which obviously depend upon certain elements of MENTAL ACTIVITY. Of these two words, *knowledge* and *experience*, the first seems too narrow for my purpose, the second perhaps too wide—in saying this I refer to the popular meanings. Hence I shall sometimes use both, as mutual correctives. For instance, we do not generally give the title of 'knowledge' to the practical skill of a draughtsman in drawing a good circle; yet it is assuredly a piece of geometrical 'experience', which is based directly on *sense-perception* (touch, muscular sensations, and sight), *and it may exist quite independently of language*. Now sense-perception is certainly knowledge.[2] Therefore, on this ground, I shall call such practical skill a form of geometrical knowledge, with as good reason as we call familiarity with 'Euclid' geometrical knowledge. Needless to say, the two are very different kinds of knowledge, and it is precisely the nature of this distinction that I shall be concerned with. Now we also,

[1] Amplified and properly interpreted, this truth lies at the very basis of an efficient presentation of any branch of intellectual education; it is in collecting and comparing material systematically for the purpose of clearest *definition* as the ideal of crude description that lies the chief educational value of the preliminary study of the subject.

[2] This English word seems to alternate in meaning between the German *Wissen* and *Können*, i.e. *knowledge* (in its narrower sense) and *faculty*; so that the combination *das geometrische Wissen und Können* would better describe the desired idea.

popularly, use the words *Practice* and *Theory* to mark this distinction. But these words have been so much used—and abused—in party controversies on education that they have almost become offensive to one side or the other, wherever a rational unification has not been effected. So I shall avoid them as much as possible. Nor do they really go deep enough into the question. My aim is to go behind these popular conceptions, and get, if possible, to the more definitely seizable mental activities on which 'practice' and 'theory' alike are built. We want the real psychological root of the matter, which, I fear, we shall never lay bare if we are for ever content to base our discussions on such obscure, and therefore necessarily superficial, conceptions. What, then, are the psychological activities at the basis of (i) this 'practical' geometrical *skill* of the draughtsman, the surveyor, the carpenter, the experimental physicist, the artist, the sculptor; (ii) this familiarity with geometric science (with '*theory*') of the professional mathematician? Such are the two typical problems that confront us, if (and upon this, I presume, we are all agreed) we would derive from mathematical education *a harmonious union of the excellences characteristic of both kinds of* '*knowledge*'. First, then, for our illustrations:—

(*a*) *Geometrical Experience or Knowledge in which* SENSE-PERCEPTION *is the vastly predominant element of Mental Activity: the Senses concerned being the Motor-Sense and the Sense of Touch.*

Ex. I. An example of this, so obvious as generally to escape notice, is the minute acquaintance we have with the shape and size of the mouth-cavity, obtained by almost incessant tongue movements and tongue contacts. Let but the tiniest piece be broken from a tooth and we become at once aware of it. Here we have direct apprehension, direct knowledge, of at least shape, size, position, distance, motion, surface, solid, point—to mention no other relations or qualities—through simple sense-perception. I do not mean to say that no other mental activities than sense-perceptions are involved in the process of development of this intimate knowledge: for it is certainly not so. Nevertheless, perception by the senses is undoubtedly the vastly predominating element. It might, indeed, be hard to find

any such completed body of geometrical experience that approaches more nearly to pure sense-perception, *to pure sensation as distinct from thought.* We might also select instances of spacial experience due to sensations consequent on limb-and-body movements. It is worth remembering, too—for educational purposes—that the development of the senses of muscular movement and of touch appears to have long preceded that of sight. Hence we can dimly understand the fundamental importance of movement and touch in evolving spacial experience in consciousness. This fundamental position occupied by touch and movement indicates clearly the wisdom of assigning a correspondingly fundamental position to them in geometrical education. The admirable skill of a good draughtsman is largely based on such sense-perceptions, as well as on sight. An inherent weakness of the old geometrical education was the failure to appeal sufficiently to movement and touch—far too much reliance was placed on *sight* for the sensuous basis of geometry. On these grounds, amongst others, we sympathize fully with those educationists who emphasize the importance of *doing* [1] in addition to merely *seeing* and *thinking.*

The high degree of development of spacial knowledge, extending to a capacity for assimilating Newton's *Principia*, that can be attained without the sense of sight is remarkably shown in the case of Saunderson, the blind mathematician, whom I have had occasion to mention before. To a less degree, but still sufficiently striking, we see the same truth exemplified in any institution for the blind.

(*b*) *Geometrical Knowledge in which* SENSE-PERCEPTION *is vastly predominant: the Sense concerned being here mainly the Sense of Sight.* Through sight, also, we become, of course, further familiar with shape, size, surface, &c.; and it is to this sense that, once infancy is passed, we appear mainly to owe the further development of geometrical experience. Waiving very obvious contributions of the

[1] Pickel well says, in his much-to-be-recommended *Geometrie der Volkschule* that the first applications of arithmetic to geometry should not be of this kind: 'The radius of a globe is 6 in. Find its surface and volume?' but: 'Here on the table lies a globe; come and measure its surface and volume.' The first question may be a discipline for sight and thought, it may also be a mere appeal to rote-work; the second necessarily appeals to sight, touch, and thought, and is the only real test of practical mastery of the idea.

sight-sense to knowledge, I select a few instances of a less trite character in which sense-perception is clearly predominant.

Ex. II. On any surface draw any enclosed figure: choose any two points on the surface, one (A) within, the other (B) without, the closed figure. Then we *see* that any continuous line whatsoever drawn on the surface to join these two points *must cut the closed figure in at least one point* (P).[1]

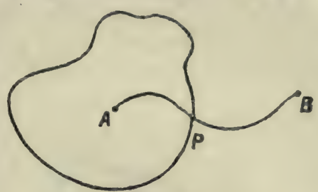


FIG. 93.

Ex. III. Take a pair of gloves, exactly similar to sight in all respects, with the exception that one is right-handed and the other is left-handed. Then we *see* that it is impossible to *fit* one upon the other without turning one of them *inside out*.

Ex. IV. Sometimes a geometrical problem which unaided *thought* finds extremely difficult to solve becomes obvious at once by appeal to experiment through the aid of the *senses*. I select an instance from what is known as the science of Topology,[2] or 'analysis situs'.

Take a rectangular strip of paper (for convenience, say

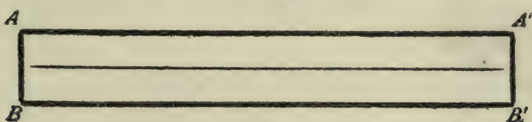


FIG. 94.

1 ft. by 1 in.). Draw a central line along it. (See Fig. 94.) Stick together[3] the two ends, with A' on A, and B' on B,

[1] To *describe* this fact, one must, of course, use *language* (i. e. one must use *conception* or *thought*); but the description or statement of the fact is something quite distinct from the fact itself as *apprehended by the senses*. A like remark is to be understood throughout. In this connexion it would be most interesting to get a true conception of the geometry of the *deaf and dumb*: we are sure of one fact—that it must be very considerable; and probably it is mainly sense-perception.

[2] Listing's name, to whom we owe the classical treatise on the subject. This is also a most interesting branch of geometry, in many respects very simple, highly *sensuous*, and quite neglected in education. (See Chaps. IX, &c.)

[3] A piece of stamp-edging is handy.

and so with $A'B'$ coinciding with AB, thus forming an ordinary closed band, of which (to clarify the instructions for a subsequent experiment, by way of contrast) we add a rough drawing (Fig. 95). Now insert the point of a pair of scissors anywhere through the centre-line (which now forms a closed curve: we have not inserted it in Figs. 95, 96, 97) and cut the band all round, right through this centre-line. In this very simple case, *thought* alone (though of course based on the memory of sense-perception), without need for experiment, would have foretold us the result—*two separate bands*.

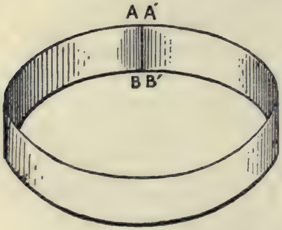


FIG. 95.

However, now select a similar strip of paper. Holding one end (say AB) fixed, give the rest a half-twist, so that it appears as in Fig. 96 when laid flat upon the table. Now raise either end and place it upon the other, so that A falls on B', and B on A'. Finally, stick the two ends together, thus forming a closed band which, placed upright on

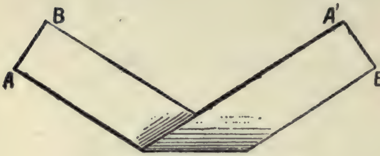


FIG. 96.

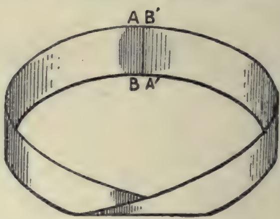


FIG. 97.

a table, appears as in Fig. 97 above. (By the way, to give a definition of this figure is an excellent exercise in description.)

The problem propounded is: What kind of figure will *now* be produced if, as before, we cut the band along the central line? [1]

[1] The crease made when flattening the band on the table (Figs. 96 and 97) must be opened out, to cut entirely along the length of the centre line.

Those who are unacquainted with the geometrical property here in question will, I think, find it a hard task to *think* out, purely from their present knowledge of space, the form of the resulting figure with detailed precision.

Make the experiment in thought: then with scissors and paper. We obtain *one continuous band*, on which are *two whole twists.*

If *this* band, again, be cut along its centre-line, what will be the resulting figure? *Two interlaced bands*, on each of which are two half twists in opposite directions, but not so as to neutralize each other—forming, in fact, a kind of 8.

Cut each again, and we obtain four bands interlaced and twisted in a complicated manner. A fourth cut (along each of the four) produces a network of eight bands so complicated that we quickly become confused with the mass of detail thus presented by sense.

At this point so admirable an occasion is presented for emphasizing the function of thought that I seize it, at the risk of some discontinuity in the presentation. When the data given to sense-perception become so complex as to cause confusion and obscurity, thought-activity takes up and continues the task of realizing the given, by processes of classification, definition, syllogism, &c. In the present case, to realize the resulting figures, after two or three cuts, the appeal to mere sense-perception rapidly grows more and more hopeless. The mind has now but one method left—to retrace its experience, *based on these perceptions*, to the beginning, and *create some kind of conceptual symbolism* capable of dealing mechanically with the complexity of the phenomena presented, or capable of being presented, with a constantly increasing number of cuts. Thus can thought, while based on sense-perception, actually advance immensely beyond the capacities of mere sense-perception.

From this example is seen how, with problems of a certain degree of complexity, an appeal to sense-perception is successful and much simpler than thought. Nevertheless, ultimately, as the complexity oversteps a certain sensibly felt limit, which varies with each individual, the need for thought becomes apparent. *It is just at this limit that the appropriate thought-machinery of abstract conceptions should be introduced to the pupil by the teacher; for the gradually developed interest and curiosity of the mind reach*

a maximum at this very point, where mere sense-perception has done its utmost. And it is precisely in such cases where the true function of thought in such processes is realized, in economizing mental activity,[1] or, to state the fact in simpler words, in saving mental labour. To sum up : The introduction of abstract, conceptual machinery of thought into education—as contrasted with sense-perception—is then, and only then, justified when the need for it is distinctly felt and the capacity for it sufficiently developed. A more thorough and accurate statement of this principle will be developed later.

(c) *Geometrical Knowledge wherein* SENSE-PERCEPTION *is predominant, but, compared with the above, decreasingly so.*

Ex. V. Draw any particular triangle ; measure each angle. Add the three results. The sum is, approximately, 180°.[2] Repeat the experiment with many kinds of triangles. It is found that the sum remains approximately constant throughout.[2]

Ex. VI. Take an empty cylinder and a graduated empty unit-cube. With the cube fill the cylinder with water. Note how many fillings and fractions of a filling of the cube are required. Repeat the experiment for a variety of cylinders. Tabulate the numbers obtained in a column ; in a second column tabulate the measures of the areas of the cylinders' bases ; in a third the heights (all, say, in feet —linear, square, or cubic). It is found that each number in the first column is approximately the product of the two corresponding numbers in the other columns.

Respecting the scope of Examples V and VI, some remarks appear advisable in view of very common misapprehension concerning them, more especially among teachers of elementary geometry. The evidence furnished by such experiments as V and VI, based directly as they are on sense-perception dealing with concrete figures which do not rigorously satisfy the scientific definitions of them, can establish neither (i) the absolute equality of the relationship in any particular case, nor (ii), still less, its

[1] The creation of this phrase and the elaboration of this truth appear to be due to Ernst Mach. See his *Die Mechanik in ihrer Entwickelung*—an epoch-making book, which science teachers should study. There is an English translation (Chicago : Open Court Publishing Co.).

[2] Inaccurate pupils sometimes get it as much as 190°.

universal truth for the corresponding figures of all shapes and sizes. For, by mere sense-perception of a particular triangle, for all we know, the result may depend on some individual peculiarity of that triangle. Even with many triangles, the result may (though less probably so, we grant) depend on some peculiarity common to those measured, but not possessed by all triangles (infinite in number). To establish these absolutely (though, of course, hypothetically[1] also), appeal must be made to the machinery of strict definition, postulate, and syllogism—in a word, to highly abstract *thought*—such as we get in Euclid. It is quite beyond the power of mere sense-perception, which can supply only empirical or experimental 'proof'. The limit of its validity in geometry is strictly confined to the actual particular cases investigated; no general conclusion is deducible, however numerous the cases may be. Hence it is false logic and vicious method to present such evidence to young pupils as universal proof of the absolute accuracy of the formulae.[2]

Nevertheless, though such evidence cannot aspire to universality and absolute rigour, its rôle in geometrical education is of great and fundamental importance; for (i) it is the necessary preliminary to the efficient grasp of absolute (though hypothetical) proof in the scientific treatment of geometry; (ii) it serves to awaken interest by suggesting the existence of some general truth; whereby (iii) the inventive powers of the pupils are stimulated to activity, both (1) to discover the empirical quantitative relation, and (2) to seize the conditions to be fulfilled by a scientific treatment.

Summing up, *no theoretical deduction from an experiment is to be drawn which the nature of the evidence does not warrant.*

(*d*) *Geometrical Knowledge used to establish Arithmetical*

[1] I have previously considered the question: How best can a transition be made between such *empirical* truth and such *absolute, though hypothetical*, proof as we get in Euclid, or any equivalent of Euclid? The question is of extreme importance.

[2] I noted many examples of this in the educational exhibits of the Paris Exhibition, 1900. The evidence was very clear; for both the teachers' syllabus and the pupils' work were open for inspection. It were invidious to name the countries, when, assuredly, in England it is still (1921) common enough.

Truth, wherein SENSE-PERCEPTION *and* THOUGHT *are, approximately, equally conspicuous.*

Ex. VII. Consider this group of dots. Selecting *rows* (horizontal), we read it as 4, 4, 1, 1, 1, 1. Selecting *columns* (vertical), we read it as 6, 2, 2, 2. Hence we conclude, *without the need for counting the total*, that these *two arithmetical totals are equal, whatever that total may turn out to be*; for we have dealt with the very same group of objects merely from two points of view.

Such a proof of equality is quite distinct from that given by actual successive addition of the parts, thus:

$$\begin{aligned} 4+4+1+1+1+1 &= 12, \\ 6+2+2+2 &= 12, \end{aligned}$$

whereby we finally arrive at the same total for each.

Now each of these groups of numbers is called a *partition* of the integer 12, where, then, by *partition* is meant the decomposition of an integer into a sum of (smaller) integers. However, our introduction of geometry here can establish considerably more than this. For clearly the figure can be legitimately used to establish the *general* truth: For every given partition of an integer we can at once obtain another, by exchanging rows for columns in the geometrical diagram representative of the given partition.

The precise relationship of the two partitions above obtained is seen, from the figure, to be: To a partition of 12 into 6 parts the greatest of which is 4, there corresponds another partition into 4 parts the greatest of which is 6.

Moreover, our figure warrants a generalization of this for any integer, so that finally we arrive at the otherwise by no means obvious universal truth: 'To every partition of an integer N into P parts the greatest of which is Q, there

corresponds another partition into Q parts the greatest of which is P.'[1]

Here *sense-perception* supplies the basis of the method, and the generalizing function of *thought* leads to the statement of the truth as universal; the total result being a so-called *intuitive* demonstration. It is useful to contrast the procedure here with that employed in Examples V and VI: in one we get *universal* truth, in the other only *empirical* truth. A rigorous analysis of the distinctions between the two methods might be a tough task, but we are certainly warranted in attributing the universality of VII to an activity of *thought*, relative to the degree of activity of *sense-perception*, much more predominant than was the case in V and VI. I trust it will be observed that the arrangement of these examples is in the direction of decreasing *sensuous* activity and increasing *thought*-activity.

(*e*) *Geometrical Knowledge wherein* THOUGHT, i.e. *Conceptual Construction, is vastly predominant.*

Ex. VIII. For a very familiar example of such, one has merely to cite Euclid's *Elements* (or any modern equivalent, such as Legendre's). Here the whole data due to *sense-perception* are (or were supposed to be[2]) placed at the very fore-front, being embodied in axiom, postulate, and definition—the definition being a more or less conventional construction of the object of thought—whereupon the subsequent process is abstract, rigorously logical deduction of the inter-related properties of such pure objects of thought by the machinery of syllogism (expressed or implied), no further appeal being allowed to *sense-perception*. The total procedure leads to what is known as absolute, though hypothetical, truth. Moreover, it is to be observed that the axioms and postulates are, or are supposed to be, reduced to a minimum, thereby still further excluding the entrance of sense-perception.

Here, obviously, we have an example in which abstraction, conception, thought, is vastly predominant.

Ex. IX. Finally, consider the nature of the process by

[1] Of course, in special cases the two may be identical. Those interested in this subject, simple in its elements and appropriate for school, may be referred to Chrystal's *Algebra*, Vol. II, chapter xxxv.

[2] For this is not rigorously the case.

which *geometrical* knowledge is evolved by the medium of *algebraic* operations in analytic geometry. Here the process is perhaps as purely conceptual and abstract as is possible for the human mind in its present constitution; in fine, here the mental activity is practically pure thought-activity in which sense-perception plays apparently no appreciable part.[1] And here, too, note that the abstractness of the process offers, among these examples, the maximum difficulty. Only the ablest intellects can follow its conceptual subtleties far without frequent revisits to the sphere of sense-perception.

4. *A Fundamental Question: How does the Difficulty of grasping Geometrical Truth vary with the quantitative proportions in which two elements—Sense-perception and Thought—are combined? Briefly put: How to economize Mental Energy?*

Reviewing the above nine illustrations, in which we have presented examples of the attainment of geometrical truth which vary gradually from almost pure sense-perception to almost pure thought-construction, and in which we obviously note increasing difficulty of assimilation, we should be inclined to put forward the hypothesis: The greater the preponderance of sense-perception, the less the difficulty of the process; the greater the preponderance of thought-activity, the greater the difficulty of the process. Now the whole body of our experience of school-education in the past, and the intimate knowledge we have of the gradual development of geometrical truth by the whole human race, both alike appear at first sight to confirm strongly the broad truth of this hypothesis, which, consequently, has come with many teachers to be ranked as a central educational principle. But we venture to think that application of the principle is thoughtlessly made to many more cases than its very limited truth warrants. In fact, the more carefully the matter is discussed the more false the supposed truth appears, regarded as a fact applicable to all ages of

[1] *Sub-consciously*, sense-perception even here exerts an unsuspected but powerful influence, whose existence probably precludes the possibility of isolating completely *arithmetical* science from *spacial* experience. One cannot write a formula so simple as $a + b$ without placing a in some definite *position* (right, left, &c.) relatively to b.

the individual and to all races of humanity. Restricted to the fields of observation, from which it was mainly first derived, namely the early years of growth, and to primitive civilizations, its truth is beyond question. But this limitation curtails within very narrow limits its legitimate area of applicability. Rigorously applied in education it certainly leads to *arrested development* (as in many ill-organized kindergartens).

It appears, then, of importance to discuss the matter carefully—we mean from the point of view of geometrical study. Now, in Example IV (p. 281) we have already presented a case in which the complexity of the phenomena overtaxed the grasp of mere sense-perception. In fact, a solution, if it could be got at all, was possible only by appeal to concept, definition, abstraction, &c.—in fine, to the machinery of pure *thought*. Such facts, repeatedly occurring as they do, at once suggest the pertinent question: 'If sense-perception is really always less difficult as a means of grasping phenomena than thought, what then has been the need of thought at all in the development of geometrical science?' The mere putting of the question suffices to exhibit the untruth of the general position. For it is clear that, though a preponderance of sense-perception is more economical of mental activity up to a certain limit of complexity, beyond this a preponderance of conceptual thought is more economical. Remembering, however, that geometrical problems are being constantly presented for solution by our constantly developing sense-experience of the universe as spacial,[1] and therefore that such limits of complexity of phenomena are being constantly reached by the seeking intellect, we venture to restate the germ of truth in the popular conception of the question thus:—(i) *The development of geometrical knowledge is due to an alternating or periodic progression of two stages of mental activity, in one of which mental energy is economized by a preponderance of* SENSE-*activity, in the other by a preponderance of* THOUGHT-*activity*. Alike in quality, these two stages differ only in the degree to which one element preponderates over the other.[2] (ii) *The sharply defined, because distinctly felt,*

[1] Whether in child, youth, adult, or race.

[2] In *extreme* cases one element may for a time occupy the consciousness so completely as almost to exclude the other: e.g. compare the con-

limits of complexity of SENSE-*presented phenomena at which the second stage of mental activity arises vary* (1) *in different individuals*, (2) *in the same individual at different ages*, (3) *in different races*, (4) *in the same race at different epochs. Observation only can give these limits.*

Having now presented the general nature of the problem, and touched on a broad solution, I proceed to a more careful discussion and a more detailed solution. Some repetition here is unavoidable on the lines of presentation I have selected; this will have its advantages in a subject so fruitful of misinterpretation.

5. *Geometry is the Resultant of* Sense-perception *and* Abstract Thought.

Alike in its historical development and in its psychological aspect, geometrical knowledge [1] appears to be an indissoluble combination of (i) *perception of the world as spacial by the senses*, and (ii) *internal construction by (abstract) thought.*

By internal thought construction I mean the mental activity involved in judgement, classification, description, and its more precise form, definition, syllogism, generalization, explanation—in a word, any kind of inference, by analytic and synthetic processes, from the simplest judgement to the most complex analysis and synthesis.

To perception by the senses, on the other hand, we owe the *materials* or *data* which, subjected by the will to the above machinery of inference, ultimately produce geometric science. Through sense-perception—and thus only—do we apprehend points, lines, surfaces, and solids; depth, length, and breadth; sizes and shapes in their countless variety; continuity, and an infinity of other spacial characteristics of material objects. Just here a misapprehension is to be feared. Of none of these qualities as scientifically, or, let me say, mathematically, defined, does sense-perception ever make us cognizant. The mathematically defined 'point' has no dimensions; the 'line' of mathematics has no breadth; the continuity of surface and line, for mathematics, involves the concept of infinite divisibility, not to

sciousness (*qua* space-concerned) of a baby with that of Lagrange creating the abstract processes of the *mécanique analytique*.

[1] To avoid complexities, I confine my present remarks to geometry.

mention other more subtle characteristics. They could, therefore, none of them be presented to the senses; they exist, in fact, only for *thought*, in the form of self-consistent *definitions*.

Sense-perception, then, is necessary for geometrical science, but not sufficient; it is only upon (more or less conventional) *definition* of sense-presented spacial qualities that we can erect a science of geometry in which *universal* truths are developed. When we say that the angles of every triangle exactly amount together to two right angles, we are aware that this is only approximately so of any concretely drawn triangle, whose angles will, in general, give a sum differing from this, for every concretely existing triangle is unique, has individuality. The general statement is true only of the triangle as defined, i.e. the ideal triangle. But definition arises from description; hence the key to the development of geometrical science lies here. I have emphasized this point repeatedly. I simply add that it is in the attempt to manufacture this key—to create the definition from crude description—by his own efforts, that the first important step in science is made by the pupil.

But to build up our *science* we need *axioms and postulates* as well as definitions. How and when are these to be developed into consciousness? What is their function in geometrical education? Here, at least, one valid negative principle our experience warrants us in formulating: the procedure of Euclid (Legendre, &c.) is *not* to be imitated, in which all the axioms and postulates are dogmatically enunciated at the start,[1] embodying thus once for all the experience of the senses, and then gaily proceeding on purely deductive lines, *thus giving no more 'innings' to sense-perception.* For the gradual conscious development of postulates and axioms forms an intellectual discipline that is quite indispensable for bridging the otherwise existing chasm between empirical observation and conceptual science—no less so than the growth of definition from crude description. In brief, the *extremely gradual genesis of geometrical science from empirical observation and measurement* forms the very first genuine exemplar of logic for the growing intelligence.

[1] e.g. in the postulate: 'Two intersecting straight lines cannot both be parallel to a third straight line', or 'Two straight lines cannot enclose a space', and so on.

But these more special questions I leave, to discuss them subsequently by the aid of detailed treatment of actual examples.

Here I am rather concerned with the general problem of the educational discipline afforded by geometrical knowledge in its unified aspects both of SCIENCE and of ART.

6. *Reference Table of Parallel Pairs of Descriptive Terms concerned with the Nature of Geometrical Knowledge.*

To return, for the sake of emphasizing now this, now that, aspect of the question—though occasionally at some expense of logical rigour—I shall henceforth refer to these two contrasted aspects of geometrical knowledge by one or other pair of the following terms :—

TABLE.

The Element of Knowledge due to Mental Activity INTERNALLY *stimulated.*	*The Element of Knowledge due to Mental Activity* EXTERNALLY *stimulated.*
Thought-construction . .	Sensual receptivity.
Conceptual element . . .	Perceptual element (or Sense-perception).
Abstract element	Concrete element.
Scientific element	Empirical element.

These pairs of contrasted elements are not, of course, by any means the equivalents of each other for all psychological purposes; but they will serve to emphasize a felt distinction which no single pair of terms is altogether adequate to describe. Hence the justification of our table. Moreover, we have to remember throughout that such a division of the two elements is probably never effected in the actual process of development of geometrical knowledge. Now one element, now the other, preponderates ; together they always appear to combine into one intelligible unity of experience. Nevertheless, just as we can reason justly about the shape of an object apart from its colour, though the two are never dissociated in ocular experience, so can we, for our educational thesis, justly separate these two elements of mental activity.

7. *The Proportions in which the two Elements are united.*

Now the central truth for mathematical education, so far as intellectual, is, from the psychological standpoint, this :

These two elements of geometrical knowledge may be, and are, united in the most varied proportions ; all gradations are possible from almost pure sense-perception to almost pure conceptual construction. It is probably this one fact of experience, patent to the roughest analysis, that forms at once our ultimate justification for abstractly sundering these two elements in psychological science, and the inducement to still more refined analysis of them.

8. *Hence the two Fundamental Questions for Mathematical Education.*

One has now only to apply this psychological truth to the educational problem in hand to be in a position to state at once the two fundamental questions to which a practical theory of education must find answers : (i) *In what proportions are these two elements—the concrete and the abstract—combined in any given piece or branch of mathematical experience ?* (ii) *What is the present capacity of any given individual, at any given age, for grasping and mastering and applying the knowledge in which these two fundamental elements are so combined ?* Speaking elliptically, though not, I think, obscurely, one may say that the task of education is to select from (i) the material appropriate to (ii). For clearly the particular mathematical experience which forms the material of the educational process must at every age, both in quantity and in quality, be appropriate to the present capacity of the individual who is expected to assimilate it.

CHAPTER XXI

PHYSIOLOGICAL CONSIDERATIONS

Concepts.

Let us further briefly consider the matter from a physical and especially a physiological aspect. At once we note a profound difference between these two kinds of mental activity—sensation and thought. Normally, the sensation (sight, touch, &c.) is the result of a stimulus due to *external* bodies acting on nerves at the *surface* of the body. This we express briefly by saying that sensations are *peripherally* excited. In profound contrast, our ideas, our thoughts, are, directly at least, the result of *internal* stimulation of nervous substance *within* the body, i.e. ideas are *centrally* excited. Now this distinction of place of immediate origin is accompanied by another distinction which, for education, is far-reaching. For, contrast the vividness of a sight sensation—the sky, an illustration in a book—with the *memory* of it after you close your eyes. Contrast the ease with which you can grasp a so-called ' ocular ' or, perhaps, a ' tactual ' demonstration in solid geometry with the sometimes insuperable difficulty of grasping the same truth with the help of thought alone. We must, in fact, admit—at least in mathematics, with other experience I am not here concerned—that, broadly speaking, of the two activities concerned in the development of knowledge, the peripherally excited activities of mind (i.e. sensations of the external world) are immensely more *vivid* and more *interesting* than centrally-excited activities of the mind, i.e. ideas. Of course, this statement needs modification for different minds and different periods of life, but the broad truth is unquestionable, and the younger the individual the more accurate and applicable is the truth. For as ideas are built upon sensations, if you fail to supply a wealth of material for sensation —e.g. with counters, bricks, paper-folding, practical survey-

ing and mensuration, in geometrical drawing, and so on —but few *ideas* will get formed at all, the inner mental life of connected thoughts will be scanty, and consequently vividness and interest will be lacking. Concepts or ideas are mere empty words in any mind in which they do not 'awaken a large group of well-ordered sensations or sense-impressions'.

A fundamental function of the teacher is the harmonizing of 'practice' and 'theory', or the concrete and abstract, by a combination of the two aspects appropriate to each age of the pupil.

Though we are able, in analysis, thus to separate out the two broad elements of mental activity involved in any given piece of mathematical knowledge, it may be doubted if, in actual experience, they ever *are* separated. The truth appears to be that even in early infancy[1] the two elements have become indissolubly united.

Even in the extreme and injurious forms of kindergarten teaching, where an appeal to the concrete and so-called practical is the sole aim of the teacher, and where sense-perception only is supposed to be trained, it is clear that a certain modicum of 'ideas' (concepts, abstraction) is indispensable and implicitly introduced. By 'ideas', here, I do not necessarily imply language. Ideas may and do exist without language. With this and the closely allied facts of sub-conscious mental life I propose to deal subsequently. Here, however, the following observation (for which I am indebted to Mr. Orr, one of my teacher-students at Sunderland) will be of value and interest. Mr. Orr blew out a lighted match in front of a baby of seven months; the baby at once looked down at the floor! The interpretation offered by Mr. Orr—doubtless the correct one—is that the baby had frequently noted that when things disappeared they fell to the ground: seeing the flame disappear it therefore also looked to the ground for it. Here, apparently, is an inference, without language, at seven months of age, and an inference of a rather complex kind.

At the other extreme, in the highest flights of the pure mathematician, where the mental activity is commonly

[1] Somewhat later, say at about 4 years old, the child capable of grasping and applying the truth *three times two equals six* has developed already a comparatively complicated system of concepts.

believed to be purely ideal or conceptual, it will be found, on rigid analysis, that some appeal to sense-perception is constantly made. Nevertheless, common sense—with which philosophy must make it its business to agree in the main—is justified in characterizing certain kinds of training as concrete, and others, by way of contrast, as abstract, or, say, 'practical' and 'theoretical' respectively; (e.g. kindergarten teaching on the one hand, Euclid on the other). Now the truth underlying this valid distinction is, I take it, that in each kind of training both elements are united, but in vastly different proportions. It is just here that, from an intellectual standpoint, we seem to reach the educational root of the whole matter. For fear of misunderstanding, let us return again for a moment (at the risk of monotonous repetition) to our antithesis between the work of sensation and the work of thought. By the *senses*—and by these alone—we get to know points, lines, surfaces, solids, shapes, sizes, positions, &c., the one and the many, in all their endless variety—here, in fine, we have the bricks and mortar, the whole *material* in fact of the mighty building called mathematics, but where is the architect to *fashion* the building? I reply, in those activities of mind which we call memory, imagination, the generalizing impulse, in the activities underlying analysis, synthesis, attention, comparison, classification, definition, and any kind of inference. These constitute, from the very beginnings of individual knowledge, the other and equally indispensable element of mathematical experience.

But these two elements—the concrete and abstract, the sense-perception and the conceptual—may be united in enormously different proportions in different pieces of mathematical knowledge, and the capacities of child, boy, youth and adult also vary, also *in enormously different degrees*, in assimilating, mastering, and applying any given piece of knowledge.

One fundamental function of the teacher is, therefore, clearly this: that, upon the basis of accurate observation of the ratio in which these two elements of experience—sensation and thought—normally enter into the *ordinary* life of his pupil at any given age, the teacher, *in school*, should pay an amount of attention to each element reasonably in proportion to that normal ratio.

The ingredients of School experience, as regards sensation and thought, must correspond with the nature of the experience outside school, and any transition must be gradual.

It scarcely seems necessary to discuss this point seriously. Any experienced mathematical teacher, upon due reflection, will surely admit its validity. I would merely add two remarks: (*a*) the ratio or degree in which sensation and pure thought are combined at any given age depends, of course, upon the state of development, at that age, of the brain and other nervous tissue throughout the body; (*b*) any attempt to teach a piece or branch of knowledge whose nature does not correspond with such physiological capacity must inevitably result in giving the shadow without the substance—concepts prematurely presented are so much sound, nothing more.

For the child, the teacher must direct his main attention to sense-perception, though taking care not to neglect the gradual development of those powerful mental activities—the architect of experience—which I have above briefly enumerated and whose use leads to the formation of concepts. For the boy and girl an approximately equal combination of sense-perception and abstract thought appears, from experience, to be justified and most efficient. For the youth, *thought* should be the predominant partner in the building up of his experience in mathematics. But, to avoid misunderstanding after such a broadly outlined division, one must add that the passage from a *predominance* of *sense*-perception to a predominance of *thought*—that economizer of time, when timeously assimilated—will be the more effective, the more gradual, continuous, and insensible it is, throughout the whole process of education.

On the diagram (see p. 298)—and subsequently on the historical diagram (see frontispiece)—I have represented sensation (or sense-perception) by white, and the conceptual (thought, idea) by shading. If now, with the above symbolism, we represent the mathematical experience of the young child by the contents of the smallest, innermost circle—predominantly white, or sense-perception—and the most effective manner of developing his or her mental life up to manhood and womanhood, by a series of concentric, ever-growing rings, then, if the principles advocated

are sound, we shall find alternations of abstract (shading) and concrete (white) experience throughout. But there is

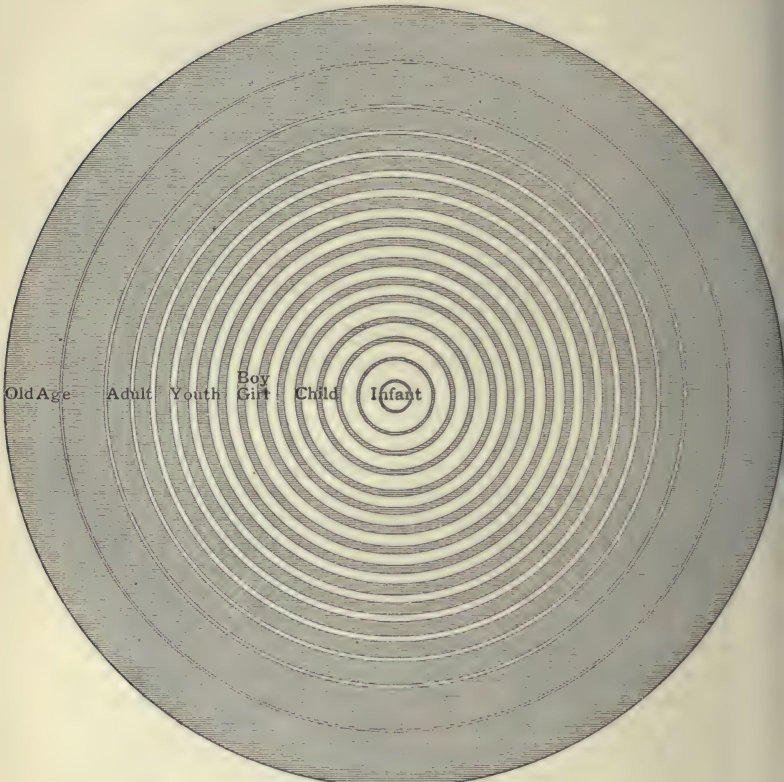


FIG. 98.

this profound difference between the start and the finish, that the predominance of one element at either end is so pronounced that a hasty observer is actually deceived into assuming that, in childhood, *all* is sense-perception (white), and that, in the adult, *all* is thought (shading).

Whence arise the two regrettable extremes, the constant battles between the advocates of 'theory' (thought) and the advocates of 'practice' (sense). Here, I venture to think, we see a possibility of reconciliation between the two schools of education. This gradual replacement of sense-perception by thought, as life advances (think of old age with its decaying senses but often bright inner life and mental activity), is an instance of a wide-embracing principle of life—the tendency to *economize* mental energy.

Concepts are the mind's tools for understanding and dominating the world around us.

It is advisable to warn the teacher not to draw from the above simple symbolism false conclusions not intended to be conveyed by it. No truth can be more than very partially symbolized thus. To avoid possible misunderstandings arising, I subjoin a few remarks on the above diagram.

(*a*) The symbolism of each ring is necessarily incomplete, if only from the fact that any state of mind, or piece of knowledge, contains both elements, though one is generally predominant.

(*b*) The rings represent, only in a *relative* sense and extremely roughly, the ratio of the ingredients entering into experience at any given moment or age. In reality, the quantity (if one may use such a description) of sense-perception, of familiarity with the world around (qua sense-perception) actually possessed by the adult is, of course, vastly greater than that possessed by the child, but any given piece of experience, in being assimilated by the child, normally contains, on the other hand, a vastly greater proportion of white to shading than is the case with the assimilation of experience by the adult. Similar statements are, clearly, applicable to the comparison of any two differing periods of life.

(*c*) The greater the wealth of concepts created by the mind in its efforts to understand and interpret its sense-impressions (i. e. to understand the actual world), the greater will ultimately be the store of well-ordered and systematized groups of sense-impressions gained. The aim of the mental machinery is to create more and more general concepts as the most efficient, the most economical tools with which to discover and systematize sense-impressions.

The infinite spiral of knowledge: new sense-experience develops concepts, and concepts develop new sense-experience.

Concepts enable us to dominate the external world by understanding the real significance of the wealth of sense-impressions derived therefrom: in *this* sense, 'the understanding makes nature . . .' intelligible to us. On the other hand, think of such concepts as number, size, shape, incommensurable position, length, angle (plane or solid), limit, function, continuity, acceleration, differential, fraction, imaginary number, ratio, similarity, convergency, degree, cosine, covariant, decimal, determinant, infinity, conservation of momentum, position, factor, generalized co-ordinate, solid, scalene, conservation of energy, entropy——

I would ask the teacher of mathematics to reflect carefully over the ideas suggested by these words. According to his mastery of mathematics his ideas will be scanty or rich. Let him try some or all of the following experiments—some of them very difficult to carry out—with as many of the foregoing technical words as he has patience for:—

1. Give a definition of each phrase, and a definition of each term he therein uses in a technical sense as far back as his interest in such a lengthy pursuit takes him.

2. Give a few illustrations of their use—numerical if possible.

3. Try to recall the kind of difficulties he or his pupils encountered in the attempt to familiarize themselves with their application and significance.

4. What sense-experiences (or experiments) did he traverse in their study—from childhood onwards. What use of sight, touch, of the muscular sense, of temperature, of hearing, did he make therein? What use of *measurement* has he made? What kind of *images* (visual, tactual, aural, &c.) do the words recall? What *other* knowledge of the words does he possess independently of, or in addition to these images? To what degree and in what sense is his grasp of *algebra* independent of *geometrical* ideas?

5. What *memory*-practice in the way of specific examples did he get?

6. How often does he use these ideas now?

7. How does his grasp of their significance vary with (*a*) the frequency with which he has employed them (i) in

the past, (ii) now; (*b*) the quantity of experiment in sense-perception upon which his study of them is based?

8. What is his power of applying them to fresh problems?

9. What degree and limits of generality does he attach to each?

10. To what degree are his knowledge of them and his ability to use them correctly, according to the more or less conventional laws to which, in operation, they are subject, *merely* rote-memory and superficial?

11. Does he consider he has an *exhaustive* knowledge of the significance of any one of them, or does he hold that there is no limit to the continued use and development either of them or of his own knowledge of them?

12. What success would he get if he put such problems as the following to his different classes of pupils?

(i) Here I have drawn a rectilinear polygon—or even a triangle—upon the board. Come and find how many square inches it contains.

(ii) Here is a sphere, or a football. Find as closely as you can the number of cubic inches it contains.

(iii) Come and show me how to measure the height of this tree, or this church-tower.

(iv) Find, to within 5 p. c. of accuracy, the positive real roots of these equations:—

$$x^2+3x = 1;$$
$$x^4+x^3 = 3+3x+2x^2.$$

(v) What is the connexion between Euclid II. 14 and the solution of a quadratic equation? Use the construction to solve approximately the equation $x^2-6x+4 = 0$.

(vi) I hold this chair fixedly up in my hands. What measurements—and as few as possible—would you make of its present position in the room, so that, when I replace it upon the floor, you could recover the exact position it now occupies?

(vii) Here is a freely-jointed polygon of rods (with five sides). Solve, with respect to *this* body, the same kind of problem as the last.

(viii) Here is an exact model of a steam-engine. The capacity of its boiler is 150 cubic inches, and the total surface of the boiler is 160 square inches. The funnel of the

model is $\frac{1}{10}$ of the length of the funnel of the original. Find the capacity and total surface of the boiler in the original.

(ix) Here are two cardboard plane polygons. Are they similar? Use the fewest measurements you can to find out.

(x) Here are two similar polygons. How many times does the area of the one exceed that of the other?

(xi) What measurements—and as few as possible—would you make to construct a figure identical in shape and size with this? Now go into the workshop and construct it.

(xii) Explain the principles involved in your solution of these questions.

Now all these types of problems *appeal to quite fundamental and simple notions*, and can, in general, be solved by any *properly-taught* intelligent lad of thirteen or fourteen—many of them, indeed, by much younger pupils. But how many lads, under the present system of teaching mathematics, would succeed?

To awaken still further the activity of the teacher I suggest that he try the following additional questions—if necessary, returning to these pages repeatedly, till he feels he has grasped the central truths emphasized therein:—What *is* a *word*? What is common to *all* the above words and phrases? What *is* conception? What function does it perform? What is the relation between a *percept* and a *concept*? What attitude of mind does the mention of a concept awaken in us according to our familiarity with its use? At this point the thoughtful and patient teacher will be in a frame of mind prepared to grasp the deep significance for education of the psychological truths underlying this whole complicated matter.

E. Mach has described so forcibly the aspect I wish here to emphasize that I avail myself of his words:—

'The definition of a *concept*, and, when it is very familiar, even its name, is an *impluse* to some accurately determined, often complicated, critical comparative or constructive *activity*, the usual *sense-perceptive* result of which is a term or member of the concept's scope. It matters not whether the concept draws the attention only to one certain sense (as sight), or to a phase of a sense (as colour, form), or is the starting-point of a complicated action; nor whether the activity in question (chemical, anatomical, and mathematical operations) is muscular or technical, or performed wholly in

the imagination, or only intimated. The concept is to the physicist (or mathematician) what a musical note is to a piano-player. A trained physicist or mathematician reads a memoir as a musician reads a score. But just as the piano-player must first learn to move his fingers simply and collectively, before he can follow his notes without effort, so the physicist or mathematician must go through a long apprenticeship before he gains control, so to speak, of the *manifold delicate innervations of his muscles and imagination.* Think how frequently the beginner in physics or mathematics performs more, or less, than is required, or how frequently he conceives things differently from what they are! But if, after having had sufficient experience, he lights upon the phrase "coefficient of self-induction", he knows immediately what that term requires of him.

'Long and thoroughly practised *actions* . . . are thus the very kernel of concepts. In fact, positive and philosophical philology both claim to have established that all roots represent concepts and *stood originally for muscular activities alone.*'

Again, '*The "general" or "abstract" is not a content at all,*' says Professor Baldwin. '*It is an attitude, an expectation, a motor-tendency. It is the possibility of a* reaction *which will answer equally for a great many particular experiences.*' These aspects of a *concept*, emphasized by Mach and Baldwin, though not exhaustive, appear to be well worthy of serious consideration on the part of teachers.

It becomes gradually clear that the main function of '*concepts*' in the getting of knowledge is to summarize *past* experience with a view to the interpretation and mastery of *future* experience. From which it directly follows that the presentation of the 'name' without a sufficiency of the 'experience' upon which it should be gradually built is the giving of the shadow without the substance, and results in inability to *apply* the concept adequately to future problems.

While thus having emphasized mainly the debt owed by concepts to sense- and feeling-experience, we must not minimize the other aspect—to wit, the equally great debt owed by sense-experience and feeling-experience to the concept. The existence and growth of the concept, in its turn, enables man to see, hear, touch, taste, smell, and feel

to a vastly more penetrating, detailed, and yet comprehensive degree—to penetrate more deeply Nature's secrets, by rendering the senses conscious of otherwise unobserved phenomena and features. For these very concepts, these internal tools of the seeking mind, indicate to the senses *what* to look for, *where* to look, *how* to look, *when* to look, and *why*.

Has not a prince of observers (Faraday) told us that in an experiment 'the eye sees what the mind looks for'? The same truth holds of experience, as a wider experiment. The mutual stimulus is surely clear. Sense-experience develops, sharpens, and extends the sphere of applicability of concepts, while concepts develop, sharpen, and extend the sphere of applicability of the senses in the world around us. One condition of the progress of knowledge is this slow but constant development in the sharpness of the senses, under the stimulus of penetrating concepts which keep the senses ever working at the very confines and temporary limits of their capacity, in order to verify the deliverances of the seeking concept.

Mathematically expressed—the increase in reality and subtlety of concepts, in detail of observation of natural and mental phenomena, in generalization of truth, and in mastery of our environment, are all functions of each other, such that an increase in one, in general, produces an increase in each of the others.

CHAPTER XXII

THE EVOLUTION OF AXIOMS IN RACE AND INDIVIDUAL

A Suggestive Experiment.

FIRST, I relate an incident that occurred when I was teaching a little girl the elements of geometry. We had for some lessons been discussing angles. With the view of obtaining two identical paper-triangles, from one of which the pupil was to cut off the three angles, place them beside each other, and note the result, &c., I cut out, in the presence of the child, from what I took to be a double sheet of paper, a triangular shape, and obtained, not two triangles identically equal, but *three*. Of course there happened to be three sheets instead of, as I thought, two only. This accidental result led to the following

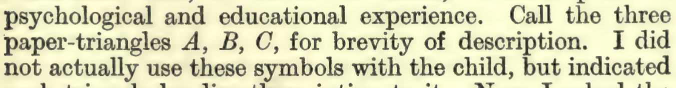


FIG. 99.

remarkable and, I venture to think, valuable piece of psychological and educational experience. Call the three paper-triangles *A, B, C,* for brevity of description. I did not actually use these symbols with the child, but indicated each triangle by directly pointing to it. Now, I asked the child to see if all three triangles were or were not exactly alike in shape and size. I had, I thought, noted from other experiences that the fact of figures all having been cut out *simultaneously* from sheets of paper in contact did not *directly* suggest the equality: hence my question. The child thereupon took up *A* and placed it over *B*, side on side, angle on angle, and found that they fitted quite

accurately, and agreed that they were exactly alike. Then the child took up C, and found, after fitting C on B, a similar result—viz. C and B were exactly alike.

I was then on the point of saying, 'You see, then, that *all three* are exactly alike,' when, to my astonishment, I saw that no such conclusion had presented itself to her mind, even from the foregoing tactual and ocular evidence, for *she lifted up* A *and* C *and began to fit one upon the other!* Here I would parenthetically remark that it is seldom wise in teaching, especially children, to *assume* even the simplest conclusions, for, however obvious to the adult, they are often, as in this case, actually beyond the logical capacity of the pupil.[1] I am aware that this maxim is familiar to teachers, but only lengthy teaching experience and intimate acquaintance with the degree of logical power of the mind at various ages can enable one to appreciate its full depth and avoid its frequent violation.

However, I prevented the child from proceeding with the actual concrete comparison of A with C, and (to test the maturity of the child's thought in these matters) said, 'Can't you tell me, *by what you have done already*, whether A and C fit each other *without actually comparing them*?' It is to be here noted that the child had been trained, in matters where she was capable of verifying, as here, to rely entirely upon her own judgement, and not to accept anything merely on the dictum of the teacher. To make certain that the question was not misunderstood, I repeated it, along with the facts already verified, in many guises, but the child frankly persisted in maintaining that she did not at all see why A and C must or should be exactly alike simply because each was exactly like B—in fine, *she would not, she said, be convinced without actually fitting* A *and* C *together!*

Seeing some subtle difficulty here, I took A and B again, marked, on each, corresponding angles and sides—ditto with B and C—so as to reduce the complex triangle to simpler elements. But without success. At length I said, 'Now, here (indicating it) is one side of A. Fit it on to the corresponding side of B.' She did so. They coincided, and she agreed that the two sides were of equal length. Then she did the same with B and C. But still the pupil main-

[1] See also Chap. IV, p. 44.

tained that she did not see clearly why the two sides of *A* and *C* *must* be equal—without actually comparing them. Then I tried the example of comparing her left shoe with a shoe of some other (imaginary) little girl, and her right shoe also with that same third shoe. I asked her to imagine that each (right and left) shoe happened to be, as proved by fitting, exactly the same length as the shoe of the other girl. But she still could not draw the conclusion I expected, viz. that in length the right shoe must equal the left.

I may add that I also tried the common mode of abstract reasoning with the child, but (as I expected) the result was a perfect blank: she could make nothing of such a purely abstract axiom as 'things equal to the same third thing are therefore equal to each other'. But, on the other hand, with the original concrete experiments she had begun to get some glimmering of the result, but was still not at all clear. At length I got a *measure*, divided into inches. Now she had often used a measure correctly and with intelligence, and repeatedly, although quite unconsciously, had, in comparing the lengths of two objects by measuring each in turn with a foot-rule (constructed by herself), made *implicit* use of the above axiom. But when I attempted either to draw into explicitness the underlying axiom or principle, or to apply the principle in a more *naked* state, as it were, than with measuring by inches, she failed to respond, completely in the first case, and partly in the second. I now, therefore, allowed her to *measure* the two sides of *A* and *C* (temporarily removing *B* altogether), and she found each to be 7½ in. Then I said, 'Now you have not superimposed, i.e. *directly* compared, *A* with *C*, and yet you can see that——?' She finished my sentence and answered my implied question by saying quickly, '*The two sides of* A *and* C *are equal*.' Then I took a piece of paper, the length of the measure (7½ in.), and showed that each side in question, in *A* and *C*, was equal to this paper, by actual superposition. Here she saw, at last, the required conclusion and, simultaneously, tolerably clearly the reason for it; but I noted she was not too confident. Remember, all this time, she had not been once allowed to superimpose *A* on *C* itself. Doubtless, too, the complexity of elements in the triangles helped to confuse her, but of this subsequently.

Finally, I tried to bring the reasoning home to her by approaching the required result as a *limit*. I took a piece of paper of which the length was $9\frac{1}{2}$ in. (though of this she was unaware). This she compared directly with B's side, and found it longer than B by 2 in. Then I said, 'How much longer is it than this side of A ?' Answer (at once): 'Also 2 in.' Here note the odd result: equality she could not easily deal with, but the intuition for dealing with *inequality* she appeared to be master of; of course, the preceding experience helped. Then I repeated the same kind of experiment, but this time with an *imaginary* slip of paper, in which the side was only 1 inch bigger (i. e. $8\frac{1}{2}$ in.) than the side of B. Successful result, as before. Then an imaginary slip with the side only half an inch bigger. Again, successful result. Finally I said, 'Now, really, as you know, the C side (reverting to the original *C triangle*) is *not any longer* than the B side. Is the C side, then, any longer than the A side? 'No!' 'Then the two must be equal?' '*Yes! I see it more clearly now.*' But I saw some slight doubt was still lingering, so, after she had repeatedly asked to be allowed to do so, I let her test the equality of A and C by *fitting them* on each other. This she eagerly did, and exhibited great astonishment that the two triangles A and C really fitted perfectly! Of course, one must remember a triangle is a complicated figure in an experiment like this, where equality involves so many different elements. But this only shows the greater need there is for caution in assumption, for, to the expert adult, a triangle is one of the simplest of figures.

Then I gave a purely imaginary example about three little boys—John, Tom, and Harry. John comes into the room with Harry, and is found, on standing up beside him, to be of the same height. Then John goes out, and Tom enters: he also is found, on standing up beside Harry, to be of the same height. *What conclusion? At first, she refuses to admit any conclusion!* So (remembering the secret of the former success) I now asked her—as another little riddle to solve—to suppose that Harry comes into the room and is *just able to stand under the mantelpiece*. Then out he goes, and in comes John, who also is *just able to stand under the mantelpiece. Any conclusion?* '*Oh, yes: Harry and John are the same height.*' Here I was apparently

successful, but not wholly so, for she quickly added, '*But you have not got three boys this time!*'

Here the lesson finished.

Subsequent reflection on the whole matter, and particularly the light revealed by her last remark, appear to warrant the statement that one great difficulty lay in her inability, out of three similar objects (whether three boys, three boots, or three triangles, &c.) to regard one of them *under a double aspect*. For one of the three objects (say Harry) has to be regarded partly under the same aspect as the other two objects (e.g. when John and *Harry* are compared, and Tom and *Harry*), and partly under a quite different aspect, namely, as a *measure of comparison* (e.g. Harry, in the above, as the measure). Finally, there is the difficulty of eliminating subsequently those qualities of this third (intermediating) object which it has in common with the other two.

This interpretation of the difficulty is strengthened by the partial or greater success of the other experiments, in which the third object is different from the other two, and used as a *measure* merely (e.g. the foot-rule, the slip of paper instead of triangle, mantelpiece, &c.). I would venture to urge the teacher to ponder carefully over this experiment, to compare it with his own experience, and to make similar experiments. I believe he will thereby learn gradually the explanation of much that has perhaps been obscure to him in the child-mind, and avoid many difficulties. It would, of course, be ridiculous to base any wide general deduction on one single experiment[1] such as the foregoing. Yet it is highly suggestive and forms a suitable introduction for some remarks and suggestions on axioms and proof, based on twenty years' teaching experience and the history of mathematics itself.

Axioms, Definitions, and Proofs: some suggestions and criticisms.

(*a*) The subject of my inquiry in this section is so complex, and has so deeply exercised for ages the intellects of many great thinkers, that I must confess that it is with great diffidence I venture to publish my own personal views on these deep-lying questions. For I recognize only too

[1] See also Chap. IV, p. 44.

vividly that in a sphere of abstract inquiry where so many powerful minds have blundered, I cannot reasonably hope to avoid similar pitfalls myself. I should therefore prefer my readers to regard these remarks as in no wise representing finality even in the very limited purview of my own development, but rather as suggestions that may stimulate others more capable and with more leisure to review and re-test many widely-accepted dogmas.

My excuses—and I trust they will be deemed sufficient—for venturing to offer even the little I have to offer are these: First, amidst much that is probably wrong, I have *something* of truth to offer which, if not new, is at least insufficiently familiar. Secondly, the whole philosophy of these questions has a very direct and serious bearing on education which, in my opinion, is as yet but little recognized. Thirdly, I have at least fulfilled the conditions without which no one has, it appears to me, reasonable right to inflict his opinions on these matters upon the public—the condition that one should have made a reasonably lengthy study of the whole subject from the four essential standpoints of philosophy, mathematical science, historical mathematics, and practical educational application. For the sake of conciseness, I shall often throw my statements into apparently dogmatic form, nor do I propose to attempt, in general, to state all the grounds on which I base my conclusions. The deepest grounds arise from the combined experience of feeling and thought slowly developed from one's historical reading, one's teaching, and the study of mathematics and general philosophy. To dig these out into the clear light even of my own consciousness would be a task certainly exceeding all the leisure I have for study and introspection, and probably in any case would prove to be beyond my capacity, for:

> 'All thought begins in *feeling*—wide
> In the great mass its base is hid,
> And narrowing up to thought, stands glorified,
> A moveless pyramid.'

Axioms.

The following quotation from Mill's *Logic* (Bk. II, chap. iii, § 3), in conjunction with the little experiment

narrated at the beginning of this chapter, forms an appropriate starting-point :—

'It is justly remarked by Dugald Stewart that (α) . . . in mathematics . . . it is by no means necessary to our seeing the conclusiveness of the proof that the axioms should be expressly adverted to. (β) When it is inferred that AB is equal to CD because each of them is equal to EF, the most uncultivated understanding, as soon as the propositions were understood, would assent to the inference, without having heard of the general truth that "things which are equal to the same thing are equal to one another".'

For convenience of reference, I have marked these two sentences as (α) and (β).

In the light of my whole experience of teaching, upon this passage I venture to remark that (α) appears to me to be true, while (β) is false. The really 'uncultivated' understanding, whether of adult or child, cannot, I believe, draw even the conclusion in (β), *in the way there presented*, at its first meetings with such reasoning (supposing, of course, 'the propositions are understood').

Psychologically the process actually traversed in such cases appears to be :—$AB = EF$ (because I have fitted them together and compared them directly[1]). Also $CD = EF$ (for a similar reason). '*Any conclusion?*' '*None that I see!*' would be the answer, in general, of the uncultivated mind.

If, of course (as will always, in the long run, happen), AB and CD are subsequently *directly* compared with each other, then of course AB is seen to be equal to CD, *but not at first by reason of the two previous measurements*. Gradually, however, with repetition of similar experience, there will be developed the observation that the *third* test or measurement is *unnecessary*, and that *it is implied in the first two*. With repeated verification, *in that particular sphere of experience* (I lay deep stress upon this limiting condition, as the kernel of the whole problem), there grows in strength, simultaneously, the *belief* that whenever $AB = EF$ and $CD = EF$, *then*, at once, $AB = CD$.

Observe, carefully, three points here :—

First, the stage (β), noted by Stewart and Mill, appears to

[1] The *statement* of the reason here for the elucidation of this discussion must not, of course, be confused with the sub-conscious, intuitive flash of insight, following on the experiment, in the pupil.

be, genetically, not the first but the *second* stage, and implies some cultivation or training of the mind. The stage (α) (recourse to expressly stated *universal axioms*) is a third and *long subsequent* stage; and this third stage is *never fully completed.* Second, the reasoning (by the (β) stage) is first, more or less unconsciously, only admitted as valid in new cases which are seen to be similar to those tested, i.e. falling within the *same sphere of experience.* Here it is the sphere of geometrical magnitudes. And even this is a generalization. For first would come the spheres of experience dealing with lengths, areas, volumes, and *then* the broader geometrical generalization. Third, experience of the validity of the same kind of reasoning in new spheres will gradually extend the application of the reasoning until, in cultivated minds, we find a strong belief in the *universal* validity of the axiom. This belief in its *universality*, however, appears to be unwarranted by a rigorous survey of the facts and processes on which it is grounded.

Weigh carefully the *essence* of the process: it appears to be this (illustrating by *types*) :—

Stage I.

(1)	This length	$A =$	that length	$C.$	}	Three independent observations.
	,,	$B =$	,,	$C.$		
	,,	$A =$	,,	$B.$		

Countless repetitions of such experiences.

Stage II.

(2)	This length	$A =$	that length	$C.$	}	Two independent observations.
	,,	$B =$	,,	$C.$		
therefore	,,	$A =$	,,	$B.$		

Similar experiences, in both stages, with other geometrical magnitudes (as areas, volumes, &c.).

Also similar experiences, simultaneously, in other spheres of equality (e.g. numbers, &c.—perhaps all ultimately quantitative).

(3)	$A = C$	applicable to geometrical magnitudes in general.
	$B = C$	
therefore	$A = B$	

$$\left.\begin{array}{ll}(4) & A = C \text{ in a certain sense} \\ & B = C \text{ in same sense} \\ \textit{therefore} & A = B \text{ in that sense}\end{array}\right\}$$ applicable to magnitudes in *many* spheres of experience.

Stage III.

(5) Conscious recognition of an *axiom* implicit in the reasoning, and *occasional though rare reference to this axiom as a ground of belief.* Formally stated, the reasoning is now this :—

$$A = C$$
$$B = C$$

And, things equalling the same thing equal each other (axiom): $\therefore\ A = B$.

(6) Belief in the *universality* of the axiom [here: 'Things equal to the same thing are equal to each other'].

Such I believe to be the main genetic processes traversed, in essence, by the *race* and by the *individual.* If so, it is clearly of deep importance for teachers to grasp their significance. Different races and different individuals in the same race will vary, of course, within wide limits, in the duration of the period of time taken to traverse each stage, but, quickly or slowly, races and individuals alike appear to traverse them all. Historically speaking, the Greeks appear as the first race to have reached Stage III (5, 6). Individuals in modern civilized times have, in general, to get well into their teens before arriving at this last stage. Previous to this ripeness of brain and training, reference to axioms appear to be not only futile, but positively injurious.

Do axioms ever become universally valid?

Our axiom, let us suppose, has at length been dug out of experience and made explicit; but are we justified in regarding any axiom as *universally* valid? Put otherwise, the question is: Has the *proof*, used in Stage III (5) above, really now attained absolute rigour? No. The strength of the axiom employed is surely neither more nor less than the strength of the whole racial experience in the particular sphere (wide as that may be) in which the axiom has been found valid; and, however wide that sphere may be, it is still *finite.* Moreover, even if, ultimately, any particular axiom should be found to be valid over *all* human experience and all systematized

knowledge, this, though admittedly the highest evidence, is still *finite*.

The *ultimate strength of truths is the strength of the old man's bundle of sticks in the fable—nothing or little when self-supporting and isolated, but great when bound together and mutually supporting each other*. I think we cannot too vividly remember and apply this great truth (which itself may be likened to the *band* around the bundle) in education. For the experience of the pupil is not the experience of the adult, and his bundle of sticks is smaller than ours in strict proportion to the degree of his mind's maturity and experience in comparison with our own. It is easy to strain or break the bundle by attempting reasoning beyond his maturity.

The Growth and Limitation of Axioms.

I shall now, as briefly as I can, attempt to show that, even with the adult and the race, however highly cultivated, axioms are, with respect to their scope and validity, constantly expanding in some directions and simultaneously becoming limited in others. Further, the genesis of axioms, in the race and in the individual, tends strongly to show that we never reach finality and universality with *definitions*, *axioms*, or *proofs*.

First, I would draw attention to the simple yet pregnant facts, well established by the labours of philology, that the life of no single word is beyond the law of development—that finality in the significance of a word is never reached so long as that word continues to be used. Further, that the significance of a word depends ultimately not merely on the context, not merely even upon the whole treatise of which the context is a part, but finally on the whole of the rest of the language—and probably, in the last subtle analysis, it ends not even there. Now, if we remember that, ultimately, in a rigorously formal sense, definitions depend upon words, axioms depend upon definitions, and proof or reasoning upon axioms and definitions, it appears to be a simple and valid corollary that *axioms, definitions, and proofs never attain finality*.

It may, indeed, be replied that this very argument—and, indeed, all arguments—assume implicitly the truth of the very axiom or principle the argument would question.

But this objection, ultimately analysed, is irrelevant, because the argument pretends to no higher degree of validity than the axioms upon which it ultimately rests. Whatever limitations may be discovered to apply to the one apply also to the other.

If there is any one principle applicable to all human reasoning, it appears to be this : *That all human reasoning that presumes to be final assumes ultimately the appearance of reasoning in a circle.* But it is appearance only, for the futility may be removed in two ways, either temporarily by a bold and dogmatic deliverance of faith at some point, or permanently by a painful *regressus ad infinitum.* The latter is the final way of real progress, which, though in its nature never-ending, may be substantial in actual amount.

Directly we recognize, then, the relativity—in the above sense—of the degree of rigour attainable, we are, I think, prepared to see its importance for education.

It is, I think, a fact that the great majority of teachers firmly believe that mathematical science is distinguished from all other sciences not so much by the difference in degree of rigour but by this—that mathematical proof is *absolutely* rigorous, while other proof is only *approximately* so. The harm this belief has worked in all grades of mathematical education is, I believe, quite incalculable. That the *rigour of the proof should be fitted to the maturity of the pupil* is an educational principle that, happily, is slowly gaining ground, though as yet recognized by but few teachers and applied by still fewer.

To return from this brief educational divergence to our main thesis. I have stated above, in general terms, evidence for the non-finality of proof and the development of axioms. It is also useful to examine the matter from a much less general standpoint.

An Axiom examined.

Let us take a few axioms and examine them. I confine myself to mathematics, but I believe the same aspect is true for any and every sphere of application of axioms of whatsoever kind.

Perhaps no axiom can well be more apparently self-

evident and simple, once its terms are understood, than this :—

The part is less than the whole.

Yet the following example shows clearly that it is a mistake to regard this as universally applicable and valid. Consider the infinite totality of the natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, ..., and conceive them arranged thus :

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...
2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, ...

so that under each of the natural numbers is placed one of the even numbers. Now it is clear (I do not propose to enter closely into the reasons or into the axioms underlying them, though that would be an interesting task) that we can always find a new even number to place below each of the natural numbers in the first row. Or, to use conventional mathematical language, we can clearly establish a one-to-one correspondence between the aggregate or totality of the *natural* numbers and that of the *even* numbers. Yet the latter 2, 4, 6, 8, ... obviously forms but a *part* of the former 1, 2, 3, 4, 5, 6, 7, 8, ... Here, then, it would surely be false to say that the part is less than the whole.

Of course it may, and doubtless will, be replied :—either that we have no right to regard an infinite collection as a totality or whole, and that, therefore, the case is irrelevant; or that totalities that are infinite are not contemplated in our axioms. To the first objection I would reply that it could only be made by one not conversant with the Infinitesimal Analysis and the immensely fertilizing function it plays in mathematical science, where the conception of infinite totals has been one of the great advances made in mathematical thought, slowly evolved from the time of the Greek school onwards. To the second objection little reply need be made, as the finiteness and consequent limitation of the sphere of application of our axiom is precisely what I wish to show. I do not propose to deal with the question whether there may not be other limitations (even as applied to finite groups), but I would suggest, in the light of the above illustration, that greater caution be used in applying to a presumably infinite universe (is our space really infinite ?) principles such as the conservation of energy, the dissipation

of energy, &c., which have been proved valid (and that not rigorously) for that part of it which falls within our experience, even though it be claimed that that part itself is infinite.

Dedekind's positive definition of Infinity.

In view of subsequent remarks on definition, I take this occasion to draw attention to Dedekind's remarkable contribution to our conception of infinity. [*Was sind und was sollen die Zahlen?* R. Dedekind, p. 17, 2nd edition. Braunschweig, 1893.] In the above argument, exception may be taken to the drawing in of infinite totals on the ground that infinity is indefinable in any way positively: it is, one may say, merely the negation of finiteness. But the following illustration will show that it is quite possible to give a positive definition and conception of infinity.

Consider the two groups:—

$$a\ b\ c\ d\ e \qquad\qquad A\ B\ C\ D\ E$$

To each element of the first group we can assign a different element of the second; thus:—

a	can be	represented by	C
b	,,	,,	B
c	,,	,,	D
d	,,	,,	E
e	,,	,,	A

Clearly, too, if for the second group we put a duplicate of $a\ b\ c\ d\ e$, so that our two groups are

$$a\ b\ c\ d\ e \qquad \text{and} \qquad a\ b\ c\ d\ e,$$

then we can say that every element of any group can be represented by a different element of that same group, e.g. $a\ b\ c\ d\ e$ by $b\ c\ d\ e\ a$ respectively. But it is also clear that the group $a\ b\ c\ d\ e$ can *not* be represented thus, by a one-to-one correspondence as we say, by any *part* of itself, e.g. by $a\ b\ c\ d$. For when we have assigned representatives to four of the elements in the original 5-group, we have still the fifth element unrepresented.

Stated briefly, we say that the group $a\ b\ c\ d\ e$ is *not* a *self-representative* system, meaning thereby that it is impossible, by a *part* of it, to represent the whole.

But, in sharp contrast, consider the totality of natural numbers, 1 2 3 4 5 6 7 ..., and the totality of even numbers, 2 4 6 .... Here (as illustrated above) it is clearly possible to represent *each single element* of the first group by a *different element* of the second, and yet 2 4 6 . . . . forms but a *part* of 1 2 3 4 . . . . . Here, then, we have, in the system of natural numbers, a system *which can be represented, in a one-to-one correspondence, by a part of itself.* Such a system may be called a *self-representative* system. The positive definition of infinity may now be stated (roughly) thus :—

A system is *infinite* when it is *self-representative*, that is, when it can be represented in a one-to-one correspondence by a *part* of itself.

Systems *not* so representable are *finite.*

Here, then, the infinite is the positive definition, and the finite the negative !

Another Axiom examined.

Consider the following statement :—

If the difference between two magnitudes can be made to differ from zero by as small a quantity as we please, then we can, without finite error in the limit, replace either by the other in any numerical calculation involving them.

Up to the time of Greek mathematics there would appear to be little doubt but that any mathematician would have admitted this principle as axiomatic. Further, from my own teaching experience, I venture to hold that few, even of highly educated people, nowadays, will deny its validity if they are unfamiliar with the principles of the Infinitesimal Calculus.

We use the principle repeatedly and with perfect safety in the ordinary calculations of life and business.

Consider its application to the following case.

Let PQR (Fig. 100) be a triangle, right-angled at Q. Divide PR into a number, n, of equal parts, Ra, ab, bc .... Draw, through a, b, c ..., parallels to PQ to meet RQ (or parallels to RQ) at α, β, γ .... Now consider one of these triangles, say, $a\beta b$. By taking a sufficient number of subdivisions of PR we can make ab, and all the other divisions, small without limit. We simultaneously thereby make $a\beta$ and βb

small without limit. Consequently we make the *difference* between the magnitude ab and the magnitude equalling the sum of $a\beta$ and $b\beta$ *small without limit.* Consequently, by our axiom above, we can in any calculation, without finite error, replace, *in the limit,* the magnitude ab by the magnitude $a\beta+\beta b$. Thus, when the *number of subdivisions is infinitely large,* and each of the triangles $Ra\alpha$, $ab\beta$, ... *becomes infinitely small,* we can replace Ra, ab, bc ... respectively by $R\alpha+\alpha a$, $a\beta+\beta b$, $b\gamma+\gamma c$, ..., without making any finite error.

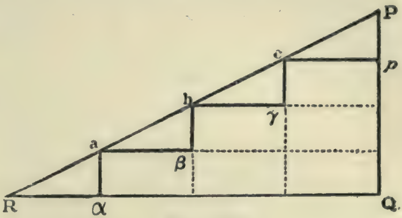


Fig. 100.

Now $\quad Ra+ab+bc\ \dots\ +cP = RP,$
and $\quad R\alpha+a\beta+b\gamma\ \dots\ +cp = RQ,$
and $\quad a\alpha+b\beta+c\gamma\ \dots\ +Pp = PQ,$
(whatever be the number of subdivisions);
∴ *if our axiom is correct,* we have :—

$$PR = RQ+QP,$$

i.e. two sides of a triangle together do *not* exceed, but equal the third!

Thus if $PQ=3$, $QR=4$, then it follows that $PR=7$, though (by Euclid I. 47) PR equals 5:—a finite mistake of 2 brought about by use of the above apparently axiomatic principle.

If some of my mathematical readers consider that no person with common sense would assent to the use of the principle in such a case, I would ask him to try the experiment, and I am tolerably confident that he will find relatively *few,* if any, non-mathematicians object to the reasoning before the conclusion is reached. Even *then* they will in general be unable to indicate the fallacy.

Moreover, a not undistinguished professor of engineering in this country, who uses the symbolism of the calculus occasionally, employed precisely this reasoning to establish the (false) result that, *in the limit,* a straight line PR coincided in length with the total zigzag path outlined thus, $R\alpha a\beta b\gamma cpP$.

Further, this same fallacy is tolerably frequent in the early history of mathematics itself.

Now contrast this case with the following:—

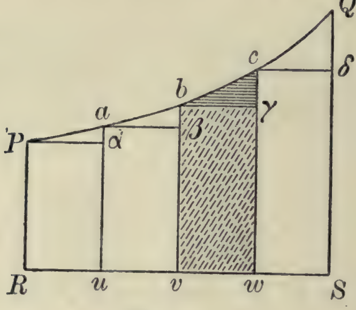


FIG. 101.

Suppose we wish to estimate the area included between the curve PQ and the verticals PR, QS, and the horizontal RS. We might proceed as follows. Divide RS into a number of, say, equal parts, at u, v, ... Through these points of division draw verticals to cut the curve at a, b, c .... Then draw horizontals through P, a, b, c .... Now consider the curve-bounded space $bcwv$. This space is bigger than the rectangle $b\gamma wv$ by the triangular-shaped, curve-bounded part $bc\gamma$. And, in the limit, by taking a sufficient number of subdivisions of the base RS, we can clearly make the three magnitudes $bcwv$, $b\gamma wv$, and their difference $bc\gamma$, *as small as we please without limit.* Consequently, again employing our assumed axiom, *in the limit,* when we take an unlimited number of subdivisions, we can make the sum of the spaces $PauR$, $abvu$, ... $cQSw$, differ from the sum of the rectangles $P\alpha uR$, $a\beta vu$, ... $c\delta Sw$, by as small a quantity as we please; i.e. *in the limit,* without finite error, we can replace the curved area $PQSR$ by the sum of the rectangles obtained as indicated.

Now this result can be shown, by a different mode of reasoning due to Newton, to be *perfectly correct* (excluding considerations of curvature alien to the case in hand), and, in fact, is the basis of a most practical and valuable method for estimating curvilinear areas.

I would now ask the reader, unfamiliar with the calculus of infinitesimals, who has had the patience to follow me thus far, to ask himself wherein consists the difference between the two illustrations.

If he does not succeed in discovering it, I would suggest a comparison between the various infinitesimal magnitudes employed, in the light of the following considerations.

The Measurement of Magnitude is Relative.

In measuring the distance between the sun and earth, an error of 1,000 miles would be very insignificant. In measuring the diameter of the earth, an error of 1 mile would be insignificant. In measuring a mile, an error of 1 inch would be insignificant. In measuring an inch, an error of one-thousandth of an inch would be insignificant, but under a powerful microscope this same thousandth of an inch would be all-important; and so on. It is clear that magnitude is *relative*, and in measurement the *importance of the error is determined by its ratio to the magnitude of the object measured.* No error is, *in itself*, either great or small, but solely with reference to the portion it is of the object measured.

Now, returning to our principle, it may be and will be argued by the non-mathematician that if an error (or difference) can be made *as small as we please without limit*, then it necessarily follows that its magnitude must be small in comparison with the object measured—*Yes*, if that object is finite; *No*, if that object is itself infinitely small. In our two illustrations the initial magnitudes or objects considered (viz.: ab and $a\beta+b\beta$ in the first case, Fig. 100, and the curvilinear element $bcwv$ and the rectangle $b\gamma wv$, Fig. 101, in the second case) are *themselves, in the limit, infinitesimal* (or infinitely small).

It becomes necessary, therefore, to ask the question:—Is the difference or error not merely small but small *in comparison with* the two objects whose difference it is?

Our answers will be: In the first case, suppose, for definiteness, that the triangle PQR (Fig. 100) has sides of 3″, 4″, 5″, and that PR is divided into a million parts. Now the small triangles, however small they become, remain similar to the original triangle PQR (viz. with sides as

3 : 4 : 5). Therefore the substitution of ab for $a\beta+b\beta$ is the substitution of $\frac{5}{1,000,000}''$ for $\frac{3+4}{1,000,000}''$, i.e. of five-millionths of an inch for seven-millionths of an inch, i.e. the error is 2/5, or a *fixed proportion*, of the object measured (ab), whatever be the number of subdivisions. Consequently the error is *never* unlimitedly small in comparison with the unlimitedly small objects themselves (ab and $a\beta+b\beta$). Our answer is therefore *No* in the first case.

In the second case, by making, say, a million subdivisions in RS (Fig. 101), and therefore a million in QT (if Pa is produced to cut QS in T), while we reduce the rectangle $b\gamma wv$ in breadth, we do not affect its height, and the same is approximately true of the curvilinear strip $bcwv$. But simultaneously we reduce the difference or error $bc\gamma$ both in breadth, $b\gamma$, and in height, $c\gamma$. The result of this is that supposing bv = 1 ft., vw = 0·000001 ft., and $b\gamma = c\gamma$, then area of $b\gamma wv$ = one-millionth of a square foot, while area of $bc\gamma$ = (approximately) one-half *of one-millionth of one-millionth* of a square foot (Figs. 101 and 102). In other words, the more we reduce the area of the rectangle, the smaller a fraction of this rectangle does the triangular area become. Further, by decreasing the area of the rectangle sufficiently we can make the difference $bc\gamma$ between this and the curvilinear portion $bcwv$ as small a fraction as we please of the rectangle $b\gamma wv$ or of the curvilinear portion $bcwv$. That is to say, in this illustration, the error is an infinitesimal part of either of the objects whose difference it is.

Fig. 102

Here, therefore, our answer is *Yes*. Consequently we must amend our original axiom now that we have discovered the existence of infinitesimals that are infinitely small even in comparison with other infinitesimals. For if, in the limit, the rectangle $b\gamma wv$ is infinitesimal, then certainly the triangular portion $bc\gamma$ is an infinitesimal part even of this infinitesimal. In the language of mathematics, if the rectangle is an infinitesimal of the first order then the triangle is an infinitesimal of the second order.

Our original axiom, therefore, requires limitation, owing to the discovery (not explicitly stated till the times of

Kepler, Descartes, Fermat, &c.) of a whole new sphere of magnitudes, viz. *infinitesimal magnitudes of different orders of smallness.* But, at the same time, it has received a new and wider sphere of application. And this will, I believe, be found to be true of *all* axioms : as science advances, their sphere of validity is constantly increasing, while their definition is made more limited and subtle.

The fundamental principle of the infinitesimal calculus.

Our axiom will now run (briefly) thus : If the difference between two magnitudes can be made as small as we please *in comparison with either of them,*[1] then, in the limit, these two magnitudes may be substituted, without error, either for the other, in any numerical calculation involving them. As thus stated (by Leibnitz and others) this forms the fundamental principle of the infinitesimal analysis.

But, to judge by the march of history, there is every reason to believe that *future ages may modify this axiom by the discovery of other species of magnitudes, at present inconceivable to mathematicians.*

The discovery (due to the demands of physical and geometrical science) of infinitesimals and their correlatives, infinites of different orders, extended the range of mathematical magnitudes as follows :—

Infinitesimals.	Finite Magnitudes.	Infinites.
i	·0000007	∞
i^2	1	∞^2
i^3	8,000,000	∞^3
$\vdots$	$\vdots$	$\vdots$
i^n	$\vdots$	∞^n

To these we have to add *absolute zero,* symbolized by 0.

[1] As now stated, observe that the difference in question need not itself be infinitely small; it may be finite or even infinitely *large,* provided the two magnitudes are themselves also infinitely large—and, in the latter case, of higher order. The axiom thereby receives a still wider sphere of applicability. Compare the original statement of the axiom, p. 318.

I have introduced these facts, so familiar to the professional mathematician, to emphasize the truths that, as the inquisition into natural science increases in depth, there correspondingly increases a subtlety of thought and a refinement of nomenclature, with greater rigour in definitions, axioms, and proofs, in the purely analytic procedure of the mathematician. But there appears to be no ground for assuming finality in any of these characteristics.

The Nature of the Evolution of Axioms.

The sphere of validity of axioms is constantly increasing in extent in some directions, is constantly being more sharply defined and limited in others, and is, in the last resort, entirely dependent on the kinds of experience which genetically gave them birth in the race. For the terms employed in their very enunciation must be *defined*, and the objects to which they apply must, ultimately, be stated as clearly and sharply as possible under the circumstances (the limitations of the engendering experiences). For example, '*Things equal to the same thing are equal to each other*.' What things? 'Magnitudes', says mathematical science, when it employs this axiom in its proofs. 'What *kind* of magnitudes?' 'Geometrical, algebraical, arithmetical, &c.' 'Define these magnitudes.' Here we begin to see (think of the history of the gradual inclusion of imaginary quantities under the conception of magnitudes and the test of their equality) that, as sciences expand, (i) the sphere of application of their underlying axioms also expands, (ii) the sphere of application of these axioms becomes limited in increasingly numerous directions (observe the axiom at the basis of the infinitesimal calculus); and (iii) this second tendency is in no sense inconsistent with the first but is actually an inevitable consequence thereof.

Hence, on reflection, we see the reason—just as in teaching we experience the *fact*—why it is unreasonable and even absurd to expect children—or anyone—to admit as universally valid, or even to use as valid in a certain limited sphere, without experience of its validity in that sphere, any axiom simply as abstractedly stated in a formal manner. For no axiom is (1) universally valid or (2) can be stated in such correct, abstract terms as to show its exact sphere of applica-

tion. How, then, can the inexperienced person know whether its use in a given sphere is included logically in the allowable sphere of validity of such axiom? The reasonably certain fact appears to be that every mathematician, and, indeed, every one who uses an axiom, never uses such axiom with assurance and confidence except in those spheres of application in which he has already on numberless occasions tested its validity by particular cases. Moreover, he did not see the significance of, nor give assent to, such axiom (except by lip-service in rote memory) until he had had numberless experiences which *implicitly* involved the use of the axiom. The history of mathematics itself is strewn with the corpses of false conclusions due to the haste and unfounded assumptions by which even eminent mathematicians have used apparently irrefragable axioms in new spheres of operation in which their validity had never been tested: spheres in which it was subsequently discovered by resulting contradictions that they were not really valid without modification (e.g. in history of infinitesimals, convergency and divergency of series, theory of equations and numbers, &c., &c.).

CHAPTER XXIII

THE GENESIS OF GEOMETRY IN THE RACE AND THE EDUCATION OF THE INDIVIDUAL[1]

MANY years have passed since Herbert Spencer, in his work on 'Education', made vigorous application of the doctrine—previously formulated by Condillac, Comte, and possibly others; foreshadowed dimly, too, by Plato—that 'the education of the child must accord, both in mode and arrangement, with the education of mankind, considered historically. In other words, the genesis of knowledge in the individual must follow the same course as the genesis of knowledge in the race.' As regards the *form* in which this doctrine is stated, no great acumen is needed to see that, in the use of the word 'must', there appears to be a confusion between the possibility or advisability of the parallelism and its necessity. The doctrine, as thus enunciated, clearly cannot rank as a principle. Its rôle is rather suggestive. How far the education of the child necessarily follows that of the race, and to what degree, assuming a tendency to the parallelism, it is advisable to modify, or even to counteract, such a tendency, these are questions suggested, but not answered, by the formula. So far as I am aware, few serious attempts have been made to indicate, with any precision, the germs of truth concealed in the doctrine when liberally interpreted and applied to mathematics.

My aim is to exhibit a parallelism between the actual mode of evolution of geometrical knowledge in the race, from the earliest times of which we have authentic historical information, and that by which the school youth can most readily and efficiently assimilate this experience. It is to

[1] Reprinted, with substantial modifications, by kind permission of the Editor from *The Journal of Education*, 1898. It has been thought desirable to reprint this essay as it brings into prominence many historical facts in mathematics not sufficiently treated in the text.

be specially remarked that I make no attempts to prove the existence of a *necessary* parallelism between the racial and the individual development of geometrical knowledge. Nor am I here concerned with the very interesting question of the almost automatic genesis of space-perceptions in the first years of infancy. What I hope to do is something quite different, viz. to show that, for educational purposes, the most effective presentation of geometry to youth, both as regards matter and spirit, is that which, in main outlines, follows the order of the historical evolution of the science.

Outline of the Inquiry.

A brief outline is desirable of the order in which I propose to deal with the inquiry. I propose to epitomize (with such fullness of detail as seems necessary for the avoidance of possible misunderstanding in the use of philosophical terms) the history of geometry from its existence as an empirical art amongst the Egyptians to its final development as a science by the Greeks, with definitions, axioms, theorems, and all the logical paraphernalia incident to a more or less perfected science. The inquiry will be dealt with from two points of view—the order of development of the *matter* of geometrical knowledge, and, of equal importance, the *spirit* in which, at each stage, it was cultivated.

The Empiricism of Egyptian Mathematics.

The earliest authentic knowledge we have of the state of geometrical knowledge before the Greeks applied their subtle intellects to its advancement is obtained from an ancient Egyptian papyrus, known as the Rhind Collection, in the British Museum, which has been deciphered only within the present generation. The date of this MS. has been variously estimated from 1700 to 1100 B.C. It is thought to be an epitome of all the mathematical knowledge at that time possessed by the Egyptians, in the persons of their priests. What kind of knowledge was this? Simply a set of *empirically discovered rules.*

In these inquiries, where a clear understanding of terms is of the first importance, it is necessary to be quite definite, a result only to be obtained by a sufficiency

of detail. What, then, precisely do I mean here by the phrase 'empirically discovered rules'? Suppose we have a rectangular surface before us—a room, a field, a figure on the blackboard—and I wish to know the magnitude of its surface.

There are but two ways of procedure—for our present purpose—and these differ *in toto*. I propose to consider one of them.

It is clear we must have a certain surface (called a unit) with whose magnitude we are familiar—itself also rectangular. I now take this unit and find, by actual trial, how many times I can lay it down on the given rectangular surface, each time in a quite new position, before I have used up all the space included within the boundary. Then, if it appears that the original surface does not contain the measuring unit an exact number of times, I may either neglect the piece left over as inconsiderable, or I may select another and smaller unit with which to make a similar series of measurements. Thus, by repeated use of smaller and smaller units, I at length arrive at one whose magnitude is so small that I cannot well make use of a smaller. There now appears to me to be no piece at all neglected. I call the measurement exact. But is it so? Certainly not; it is now correct to say, not that I have measured exactly, but that I have reached the limit of my measuring powers. The exactness is only relative, for I have merely to employ an individual with keener eyesight and more delicate manipulative capacity to obtain what *he* would doubtless, in his turn, call an exact measurement; and yet, though certainly more exact than mine, it is still clearly only a relative exactness. A little reflection, indeed, will convince one that there is no end to such an inquiry: no surface, concrete and actual, admits of absolutely exact measurement. Why not? Because, amongst other equally important reasons, we cannot define, with absolute precision, what we mean even by the *boundary* of such a surface. The very attempt lands us in a discussion of the subtlest problems of philosophy. Every succeeding generation of scientists, with deeper knowledge and better instruments, would improve on the measurement of its predecessors. From this aspect civilization appears as a function of the place of the decimal point. There is no finality.

Such measurements, then, as above described let us call experimental or empirical. Now observe that the measurement obtained with so much trouble applies only to this particular rectangular surface; it gives no information about other rectangular surfaces. Further, let us suppose that repeated measurements, by this very obvious method, of all sorts of rectangular areas, have been thus experimentally made, and the results tabulated. In addition, let the measurements of the *sides* of these rectangles be obtained in similar direct manner (by use of units of length) and let these results chance to be tabulated alongside the others. [We presume total ignorance of geometrical *science* on the part of our practical geometricians.] Finally, let us imagine some observant individual amongst them discovering, either by chance or with intentional quest, that, if he multiplies together the numbers giving the measures of the sides, he obtains, in all the cases observed, numbers very close to those measuring the areas.

It is, perhaps, interesting to observe that the discovery of such relations would appear to be almost impossible for races whose means of computation were meagre, unless the unit of length chanced to be related in some extremely obvious way to the unit of area, as, for instance, being the side of the square which is the unit of area. This remark serves to illustrate the significance of the part played by chance in the discovery of important facts, such as, doubtless, the above would be, in the history of a nation's mental development. It also serves to indicate the kind of stimulus that an appropriate study of empirical geometry should give to the inventive faculty of the child. Here, indeed, at once, we perceive a valuable educational parallelism such as we previously contemplated.

We have, then, supposed the discovery of a certain relation between sides and area. The larger the number of cases tested, the stronger would be the belief in the universal applicability of the relation. But, however many be the tests, the law is still only an empirical statement; the two groups of numbers spoken of—the numbers giving respectively the area and the product of the sides—will never exhibit more than an approximate correspondence; the equality cannot, from the nature of the case, be absolutely exact. However valuable in future

use the discovery may be, it is not a logically proved geometrical theorem, but a wide empirical induction. It ranks as a fact of experimental geometry, but forms no part of a scientific geometry. The relation might be discovered —and, indeed, appears to have been discovered—by one unversed in such abstractions as straight line, axiom, theorem, &c.

The Spirit of a Scientific Geometry.

By way of sharp contrast, let the same problem of measuring a certain rectangular surface be now proposed to a man who grasps the spirit of a scientific geometry. He is aware that, from certain arbitrarily formed definitions (of straight lines, parallels, &c.)—which, observe, are creations of the intellect worked up from sense-data, mere conceptions of the understanding—he can deductively prove from the definition of the abstract geometrical figure, termed a rectangle, that its area can be got by multiplying together the numbers measuring the lengths of its sides, provided they have a common measure, while, if they have not a common measure, a product can be obtained giving the result to any degree of precision required. Observe that incommensurability is not a property of objectively existent lines; it can logically be proved of, and therefore applied to, only ideal geometrical creations. Hence the glory of the Pythagorean school of mathematics—the creation of the theory of incommensurable magnitudes.

So far all is pure theory; the corresponding geometrical figures exist only as ideas of the man's mind; they are simply conceptions. In applying these to concrete, visible surfaces, our geometer foresees that the so-called sides of the objectively existent rectangle he wishes to measure cannot possibly be more than rough approximations to his ideally defined straight lines (e.g. they must have breadth, or he could not see them); that the surface of the rectangle, that the angles, &c., are but rough copies of his geometrical plane surface, right angles, &c. But, although this is so, such facts simply serve to exhibit the excellence of his ideal geometry for purposes of application to the concrete; since, however closely approaching straightness lines may be actually drawn, and however nearly plane surfaces may be actually made on matter, the geometrical theorems, being

based on lines defined by man's own creative thought as perfectly straight and on plane surfaces that are similarly defined as perfectly plane, &c., are thereby efficient to cope with any kind of physical measurement, however precise it may become. Indeed, the absolute precision of geometrical science ever offers an ideal towards which actual physical measurement may strive, but which it can, obviously, never reach, though ever approaching nearer. In this aspect geometry has analogy with moral law, which has neither greater nor less cogency and application to human life than geometrical theorems to the material world.

> 'No earthborn will
> Could ever trace a faultless line;
> Our truest steps are human still—
> To walk unswerving were divine.'

In the language of the mathematicians, physical measurement and geometrical are mutually asymptotic.

This distinction, which is of importance for our purpose, and frequently misapprehended, may become still clearer if we reflect what could have been the progress of physical science—in which advance appears, from one aspect, to lie ultimately in the possibility of measuring to extra decimal places (note the discovery of argon)—had geometry remained empirical. Imagine a stone geometry, in which deductions were made in terms of such points, lines, and surfaces as can be obtained on stone, with the aid of stone. How could such a geometry cope with the niceties of measurements flowing from the use of steel instruments on steel surfaces? Clearly we should here reconstruct and refine our geometry incessantly, as instruments became more precise and muscles more adaptable. Stone geometry would succeed wooden, steel geometry stone, and soon we might be floundering in the difficulties of a celluloid geometry.

All this may appear trivial, but, in view of notorious historical misapprehension of the basis of scientific geometry, the grotesque misapplication of Euclid to elementary education, and the vagueness evinced by even well-educated people concerning the nature of geometrical truth, I believe such illustrations have their use. Moreover, it is high time that teachers turned their attention to the history and philosophy of the subject they teach.

To return to the measurement of the rectangular surface. Our scientific geometrician has, we suppose, logically deduced from his conceptions of straight lines and rectangles a formula for obtaining the area of any rectangle whatsoever —i.e. a rectangle in his ideal sense of the word. Then, with the utmost precision of which he is capable, he measures the lengths of two adjacent sides of the given material rectangular surface, and, according to his formula, multiplies together these numbers, thus obtaining, in units of area, the magnitude of the given rectangle. As far as his measuring precision is reliable, so far can he trust his result; the applicability and validity of his abstract formula he never dreams of questioning—and rightly.

Contrast between the Empirical and the Scientific.

Observe the difference between the two methods of procedure. In the first (the practical geometer's method), we start with direct, particular sense-perception and experiment, and end with a wide empirical induction, based on repeated rough measurements; in the other, the process starts with a general scientific conception (formula based on rigorous reasoning from definitions, &c.), and we end in getting, through its aid, a particular concrete result. One process leads to an experimental or empirical geometry; the other proceeds from a scientific geometry. One deals with particular facts; the other with general theorems.

I have stated above that the earliest document—the Egyptian Rhind Papyrus—respecting the geometrical knowledge of the ancients consists of the statement of the results of particular measurements, or at most of empirically discovered rules. 'The papyrus contains,' says Allman (*Greek Geometry from Thales to Euclid*), 'a complete applied mathematics, in which the measurement of figures and solids plays the principal part; there are no theorems properly so called; everything is stated in the form of problems, not in general terms, but in distinct numbers—e.g. to measure a rectangle the sides of which contain two and ten units of length; to find the surface of a circular area whose diameter is six units; to mark out in a field a right-angled triangle whose sides measure ten and four units. . . . We find also in it indications for the measurements of solids, particularly of pyramids, whole and truncated. It appears

from the above that the Egyptians had made great progress in practical geometry.'

As witnessing to the very empirical state of geometry as it existed among the Jews, Babylonians, &c., it is to be noted that they appear to have thought that the circumference of a circle is just three times the length of its diameter. Thus we read that Hiram made for Solomon 'a molten sea, ten cubits from the one brim to the other; it was around all about . . . and a line of thirty cubits did compass it round about' (1 Kings vii. 23). Even this may be too much to attribute to them. There is always a danger of reading into statements of this kind more than was originally intended, a danger due to our own vast modern mastery of the science. Possibly Solomon's architect simply found by measurement that the circumference of this particular circle measured in length three times its diameter. Perhaps he was not even aware of the general empirical truth that the circumference of every concretely drawn circle bears a fairly fixed ratio to its diameter. Still further, presumably, beyond his comprehension would be the scientific theorem that for all abstractly defined circles this ratio is absolutely fixed (and incommensurable). Incidentally here remark that, unless the idea of a possible numerical dependence of circumference on diameter (or vice versa)—the notion, in fact, of a mathematical function—already exists or is suggested by analogy from other experience, there is nothing to urge the mind towards a search for the precise measure of this dependence. Here, as elsewhere, we see only what we look for, over and above that which is obvious to all. Now, this idea that, in some definite way, the two lengths are numerically related appears to have been born with difficulty. Nor, indeed, is the notion of a mutual numerical dependence common even among modern well-educated people. Many are those who know, and can mechanically apply, the fact that 1,728 cubic inches make one cubic foot, and yet are unaware what dependence this large number has on the fact that twelve inches make one foot. A specific education fails in its due effect in such cases as these, where the bare particular fact is remembered by rote, while the valuable part of the matter (here, the idea of a function) is never assimilated. Such fundamental defects still characterize much of elementary

education. Egyptian geometry, then, the precedessor of Greek geometrical science, appears to have been practical, approximate, inductive, not scientific, deductive, exact. In one word, it was *empirical*.

Do by learning: and learn by doing.

I pass on to Greek geometry. Dr. Allman (in the work above cited) has indicated the relation in which Greek geometers stood to their Egyptian predecessors, a relation which appears to have been often misunderstood. The mixed basis of geometry—partly sense-data, partly creative thought—clearly indicates use for and training of both *hand* and *thought* in geometrical education. Philosophy has long been dissociated from the teaching of mathematics, to the great detriment, I am convinced, of the latter. Education is sure to suffer in the hands of a teacher who is not familiar to some extent with the philosophy of his subject. This brief epitome is by no means inserted to inform—philosophy cannot thus be digested in compressed tabloids—but simply to draw attention to the expediency of inspiring a love of philosophical thought in the minds of teachers. The philosophic mind is specially needed in these days of educational maxims, when the teacher is on one side advised to apply the valuable maxim: 'Learn by doing'; on another side, to rely on the equally valuable maxim: 'Do by learning'. Only the teacher with philosophic breadth of view can reconcile these two half-truths into an applicable unity of method, wherein, if doing is precedent to learning at one moment, in the next as assuredly is learning precedent to doing, education being the deliberate attempt to methodize an incessant action and reaction between these two.

Greek Geometry.

A clearer understanding of the basis of geometry prepares us to appreciate the advance in geometrical knowledge due to Greek intellect. 'The first name,' says Allman, 'which meets us in the history of Greek mathematics is that of Thales of Miletus (640–546 B.C.). . . . Thales himself was engaged in trade, is said to have resided in Egypt, and, on his return to Miletus in his old age, to have brought with him from that country the knowledge of geometry and astronomy. To the knowledge thus introduced he added the

capital creation of the geometry of lines, which was essentially abstract in its character. The only geometry known to the Egyptian priests was that of surfaces, together with a sketch of that of solids . . . obtained empirically ; Thales, on the other hand, introduced *abstract* geometry, the object of which is to establish precise *relations* between the different parts of a figure, so that some of them could be found by means of others in a manner strictly rigorous. This was a phenomenon quite new in the world, and due, in fact, to the abstract spirit of the Greeks.'

Greek Astronomy and Surveying.

'In connexion with the new impulse given to geometry, there arose with Thales, moreover, scientific astronomy, also an abstract science, and undoubtedly a Greek creation. The astronomy of the Greeks differs from that of the Orientals in this respect—that the astronomy of the latter, which is altogether concrete and empirical, consisted merely in determining the duration of some periods, or in indicating, by means of a mechanical process, the motion of the sun and planets ; whilst the astronomy of the Greeks aimed at the discovery of the geometric laws of the motions of the heavenly bodies.' Thales 'measured the Pyramids, making an observation on our shadows when they are of the same length as ourselves, and applying it to the Pyramids. . . . Thales measured the distance of vessels from the shore by a geometric process'. Note these applications to the concrete. Again, we are told by the historian Eudemus that he attempted 'some things in a more abstract manner, and some in a more intuitional or sensible manner'. Thus it is clear he would continue to employ empirical measurements to obtain approximate results, which, by the creation of definitions and the use of axioms, he would gradually replace by general truths intuitively demonstrated. From a collection of such general truths there would gradually emerge a more or less systematized body of scientific theorems. Allman attributes to Thales the discovery of the two theorems—(*a*) The sum of the three angles of a triangle is equal to two right angles ; (*b*) The sides of equi-angular triangles are proportional. (Hence the basis of the theory of *similar* figures.) Thus, from a philosophic point of view, says Allman, 'we see in these

two theorems of Thales the first type of a *natural law*—i.e. the expression of a fixed dependence between different quantities, or, in another form, the disentanglement of constancy in the midst of variety—has decisively arisen'; whilst, from a practical point of view, 'Thales furnished the first example of an application of theoretical geometry to practice, and laid the foundation of an important branch of the same—the measurement of heights and distances.'

Further development of Greek Geometry.

After Thales comes the contribution of the Pythagorean school, solidifying and augmenting the previous primitive scientific work. 'Pythagoras changed geometry into the form of a liberal science, regarding its principles in a purely abstract manner, and investigated his theorems from the immaterial and intellectual point of view.' The geometry of areas plays an important part in the work of this school (e.g. Euclid I. 47), thus exhibiting the mode of evolution from its Egyptian empirical source. Again, 'the Pythagoreans first severed geometry from the needs of practical life, and treated it as a liberal science, giving definitions, and introducing the manner of proof which has ever since been in use.' Let us carefully remember that 'one chief characteristic of the mathematical work of Pythagoras was the combination of arithmetic with geometry', culminating in the theory of proportion. 'In this respect he is fully comparable to Descartes, to whom we owe the decisive combination of algebra with geometry.' Allman says of this unifying aspect of his work: 'We are plainly in presence of not merely a great mathematician, but of a great philosopher. It has ever been so; the greatest steps in the development of mathematics have been made by philosophers.'

Of set purpose have I thus focussed attention sharply on the profound contrast between *empiricism*—the final form of geometry among the Egyptians—and *science*, its final form among the Greeks. Consequently but the briefest allusion has been made above to the fact of the actual continuity of the transformation or evolution of empirically discovered rules, through the medium of intuitive insight, into a body of scientific theorems. This aspect, however, I have fully dealt with in previous chapters.

It is enough to say here that the intermediate stages were traversed with astonishing rapidity, due to the peculiar nature and power of the Greek genius. Anticipating our final educational parallel between racial evolution and individual, we might from this piece of history draw the conclusion that, in the teaching of geometry, the passage from the empirical stage to the scientific should be rapid. Or we might argue with perhaps still greater force that the passage should be made rapid with exceptionally gifted boys only, on the ground that the Greeks were themselves an exceptionally gifted race. Careful and deliberate experiments alone can decide which view is the juster.

The Spirit in which Geometry was cultivated.

Of equal importance with the question of the historical order of development of the matter of geometrical knowledge is a consideration of the attitude of mind of the ancients towards the subject, the spirit in which at different times they cultivated geometry, as art or science or both. First we find the Egyptians employing a crude empirical geometry for architecture and land-surveying, rendered necessary by the obliteration of landmarks caused by Nile floods. These approximate rules of thumb come to the knowledge of a people of higher intellectual calibre—the Greeks. Hence there gradually emerges the vague conception of the possibility of a *science* of geometry, in which clear, abstract definitions shall refine on mere sense-perceptions, axioms peculiar to geometry combine with axioms at the base of all reasoning, and thereby the empirical laws be absorbed once for all in rigorously deduced abstract theorems. Of course the emergence of all this was very gradual; there was incessant action and reaction between the concrete and the abstract, a fact of fundamental importance for education. At length we reach a time when geometrical knowledge has assumed a more or less perfected abstract form, has become evolved into a science. We find it now in the hands of professional philosophers, who follow and value the study of it partly as an intellectual discipline, and partly out of scientific curiosity. Plato, himself a student of geometry, though apparently not a specialist therein, appears simply to express a feeling common in his time when he denounces the application of scientific geo-

metry to ‘vulgar handicraft’ as demeaning to the science. We all know the story of the inscription: ‘Let none ignorant of geometry enter my door.’ To Plato and his attitude I shall presently return.

This divorce of geometrical science from the needs of common life must not be misinterpreted as a complete sundering of the abstract from the concrete. Bearing in mind the presumed educational application of this epitome of the history of geometry, I lay great stress on the fact that ‘side by side with the development of abstract geometry by the Greeks, the practical art of geometrical drawing, which they derived originally from the Egyptians, continued to be in use.’

The ideal of Greek geometry may fairly be described as *construction, under self-imposed definite limitations.* For example, as regards problems in a plane, from the abstract side of thought the attempt was made to solve them by ultimate reference to the concepts, straight line and circle; from the concrete standpoint, all constructions were to be reduced to use of ruler and compasses only, the respective concrete embodiments of the ideal straight line and circle. In the former aspect geometry was entirely independent of mechanics, but in the latter dependent on it. But not for long can the two be separated without gravest danger of arrestment of the one as art and of the other as science. Plato himself, not dreaming apparently of the possibility of the immense stimulus geometry was in future ages to receive from the needs of the mechanical art, though wisely advocating warmly the educational claims of geometry on its purely abstract side, yet unwisely condemned, in his prejudice, its alliance with the concrete. The cultivation of the Platonic ideal and aspect of geometry was destined to advance enormously the progress of geometry. But equally fertile proved the despised alliance with mechanics and the common needs of life. Fortunately, despite Plato's great influence, Greek geometers, wisely trusting their genius, constantly overstepped those limits which Plato and others would have imposed. We find them making experiments, constructing curves as loci of points got with ruler and compasses; and, finally, when the continuous description of certain curves demanded for the solution of problems—e. g. the trisection of an angle—was seen to be impossible without an infinity of single points (out of the reach, consequently,

of ruler and compass), we find them inventing and using mechanical instruments and methods for the continuous description of these curves, precisely as a pair of compasses draws concretely a continuous circle.

In these tendencies, not to be suppressed, we recognize an affinity to the genius of Newton—'At aequatio non est,' he says, 'sed descriptio quae curvam geometricam efficit,' —and, in modern times, to Cayley's fondness for geometrical drawing and for the modelling of surfaces, and to Sylvester's interest in linkages. The condemnation of Plato's view and the admission of mechanical ideas to the sacred realm of mathematical science become decisive and final when we reach Lagrange, who expressly included mechanics (the concept now, of course, being infinitely wider) as a branch of pure mathematics.

Plutarch tells us that the strictures of Plato had, at least, the unfortunate effect of retarding for long the development of mechanics. A precisely similar error we ourselves make in the mathematical education of our scholars. This remark suggests considerations that I have elsewhere developed.

The professional Mathematician : Euclid.

Finally, we reach the foundation of the Alexandrian school of science (about 300 B.C.), where we first find in existence the full-blown professional mathematician, no longer a philosopher in the Greek sense of the word, but pursuing the science, not for general culture, but for its own sake.

Of these professional mathematicians the first, and one of the most eminent, was Euclid, who systematized on philosophic basis, with substantial additions of his own, the geometrical knowledge slowly evolved during preceding centuries. This he did in his famous 'Elements'—a text-book for students of philosophy and science in the then newly founded University of Alexandria, but no fit 'meat for babes and sucklings'. To educationists it is of the first importance to understand that this highly ambiguous word 'Elements' in the title (Euclid's *Elements of Geometry*) refers not to the rudimentary psychologic elements in the genesis of the *child's* empirical knowledge of the world around as geometrical, but to the *logical* elements that emerged finally, after centuries of effort, in *mature* minds as the ultimate outcome of a long line of philosophic abstractions (definitions, axioms,

theorems, &c.) whereby geometry was fashioned into an almost perfect science.

Educational applications.

To evoke greater interest and inquiry, I add a few detailed suggestions. Waiving the vexed question of the mode of genesis of space-perception in infancy, we come to an age, varying in different children, when under appropriate stimulation, by leading questions concerning objects presented to the senses, the child becomes capable of voluntarily directing its attention to a consideration of the *form* of such objects, to the exclusion of other properties (colour, &c.). Its stock of space-perceptions, acquired partly by painful, and partly by pleasurable, struggle with its environment, now gradually becomes transmuted, by external stimulus to its own self-activity, into a *descriptive* knowledge of form, a knowledge in which perceptions fuse together into conceptions by being attached to a descriptive name. So fertilizing is a union of language with objective embodiments of form—either useless without the other—in rendering clearer, more true and precise, the early intuitions of the child. Here, as throughout education, the teacher needs faith and tact.

Great care must be taken to avoid over-preciseness in the use of terms, thereby incurring the danger of supplying the word without *any* idea. Equally harmful, however, is the other extreme, where it is imagined that the mere examination of an object, without attention to the wonderful function of descriptive language, suffices to stimulate the creative activity of the child. The one evil is premature over-elaboration and refinement of the abstract in the formation of knowledge. A recoil from this is apt to land us in the other extreme of clogging the growth of freedom of thought, either by confusing the ideas with the very wealth of the objects to be apprehended, or by failing to bring about the emancipation of the ideas from the particular concrete embodiments from which in the first instance they sprang. This latter extreme in education entails inability, in subsequent years of life, to make effective use of the narrow and particular for the emergence of the comprehensive and general. The aim throughout the mathematical education is the *mastery* of form by eye and hand and thought.

It is neither the purely abstract thinker nor the voiceless intuition of the savage we must strive to produce, but the consciously disciplined artist, at once thinker and doer.

Measurement, and the Empirical Stage.

Gradually the child gains a store of geometrical knowledge that is clear, conscious, rational, and definite in comparison with the mental results of his previous experience, but vague, empirical, and indefinite relatively to the mastery we desire him ultimately to obtain. By appropriate stimulus the child will now be incited to a desire for more exact processes, for fuller, clearer knowledge. The idea of *measurement* waxes in importance ; simple instruments are made by the child himself—many and fertile will be the ideas thereby originated—and lengths, surfaces, and volumes yield numerical results under the potent influence of simple arithmetical ideas. Tables of such results (no measurements should be wasted ; all should contribute to final results), scanned with lively attention, give rise to new demands on arithmetic. General rules for measurements emerge, with a hint or two from the teacher what to look for, and thenceforward the joy of discovery becomes the most effective of educational agents. Geometrical knowledge, and skill in simple arithmetical computations grow hand in hand ; this mutual co-operation and assimilation of the two studies is of the highest importance. Observe, throughout these final brief remarks, the historical parallel. Let not the teacher fear to introduce ideas that, in his own education, were the last of a long line of tedious symbols and abstractions extending over years, ideas which lie at the very roots of scientific thought. Thus, in the detailed measurements of triangles of varied form, attention will be drawn to the amount of change produced in the lengths of the sides by certain changes of a definite amount in an angle, one side being fixed, and, say, one angle a right angle (an empirical right angle, at present). Here we have the germs of trigonometry without symbolism. Thus is introduced the idea of a variable magnitude, and of mutual dependence. Indeed, as Herbart (*The A B C of Sense-Perception*, 1803) well remarks, all magnitudes should, from the very start, be so taught as to be constantly considered *fluxional*. Rough measurements of the rapidity with which areas and volumes grow by

adding to their linear dimensions prepare for the future easy apprehension of a differential coefficient. Plane surfaces rolled into cylinders and cones and other shapes give access to the idea of a ruled surface; such are the surfaces the pupil's pencil is constantly describing in space, as it is handled. And so on.

Noteworthy, as historical parallel, is the attempt of the Greek geometers to square the circle. They attempted to exhaust the circle by means of inscribed and circumscribed polygons with a continually increasing number of sides. Here we find the germs of the infinitesimal calculus, crude and empirical at first, subsequently developing into a rigorous deductive process (the method of exhaustion), and, finally, after centuries of laborious thought, perfected by the labours of Newton, Leibnitz, and others. Very obvious is the bearing of this on education.

The gradual passage to Science.

So far, in the pupil's education, we have assumed that all has been approximate, empirical. That the area of a concrete triangle is practically half the base into the height is, as yet, simply a wide induction. Nevertheless, but little additional stimulus is needed to convert such empirical facts into universal truths. When attention is drawn to the fact that no lines actually visible can be drawn without breadth, and that greater precision is attainable in our measurements the better our instruments and the finer-drawn our figures, the mind is fit for the discovery of comparatively rigorous definitions and of gradually systematized sets of general truths intuitively demonstrated. Whence the path is easy to a single body of scientific theorems—such as is presented in Euclid. Here, again, lies danger of an extreme. Assuredly it is a fundamental error, in school education, even when the ideas of definitions and theorems have grown familiar, to have complete divorce between the concrete and abstract. While in no whit deviating ultimately from a rigorous use of certain terms and deductively stated proofs of certain theorems, a philosophical teacher will continuously make effective use of the fact that, at every stage of scientific mastery by the pupil, there looms certain material of knowledge which can best be first assimilated *empirically*, and should only gradually be subjected to the stricter demands of exact, abstract reasoning.

Turning to history, we find that never without detriment to pure science has the abstract been long divorced from the concrete. Modern educational experience amply exhibits the pernicious effects of restricting teaching to the purely abstract. It is not long ago since Euclid was memorized by rote! The fact that all measurement of nature is necessarily approximate, never exact, is a truth that appears to have been almost completely ignored in mathematical education, fundamentally relevant to the matter as the truth obviously is. Approximations, concrete applications of pure theory, should occupy throughout the educational curriculum a fundamental place. It is clearly possible to present such practical problems that the very effort to attain a solution leads to the demand for still higher and fuller theoretical knowledge. Let us here employ to the fullest that principle of all mastery: 'Studies perfect nature, and are perfected by experience.'

If this criticism is valid, then we soon become convinced that the isolation that still too often exists between geometry theoretical, geometry practical, arithmetic, algebra, &c., is radically vicious. 'Arithmetic is one thing, algebra another, theoretical geometry a third, practical geometry a fourth, and so on. We learn them from different books at different hours, and often from different masters. We are ignorant of their relation and mutual helpfulness.' Such, doubtless, is the attitude of many a school youth when attention is directed to the question. Yet, what is the worth of all these studies unless every conception, finding its appropriate place in the scheme of all the rest of our knowledge, helps to a more clear, unified mastery of facts? *Juxtaposition of subjects in the curriculum does not imply harmonious assimilation of them by the mind of the pupil.* Without any resulting confusion, all these branches of mathematical study can be commingled and become materially helpful, so that the mind sees its mathematical conceptions and processes in the light of a beautiful, well-ordered, and powerful *whole*, instead of a thing of shreds and patches.

The insistence, in elementary teaching, upon a comparatively few ideas ultimately leads to inability to grasp new ideas when they are encountered suddenly in the higher branches, clothed gorgeously in strange symbolism. The remedy for this is to keep the invention ever at work, and

the assimilative function fresh and vigorous, by constantly bringing down for discussion and simple application into the very elements those fruitful and great *ideas* that certainly demand ultimately for deeper treatment a special symbolism for themselves, but which are relatively simple in inception when divested of such symbolism. The plotting of curves, modelling of surfaces, with the concomitant ideas of analytical geometry (plane and solid), the fundamental ideas of the calculus (differential and integral) through approximations, the plentiful implicit use of axioms (not restricting the science to a minimum of axioms, with resulting tediousness and great loss of power)—all such conceptions it is desirable to create as soon as the interest is sure to be awakened in them.

Here, again, the teacher must be inspired with knowledge, not only of those higher branches, but of their gradual historical evolution. Seeds of thought must be planted long before they grow to perfection and ripeness. Above all must he have faith in the intelligence of his pupil and the great future in store for it under the guidance and stimulus of sympathetic teaching.

Of course, in introducing these ideas of mathematics so much earlier than usual, we must not make the mistake (which would be identical with that previously perpetrated in commencing geometrical education with abstract Euclid) of attempting to present them in completed abstract form —an attempt certain to result in dire failure. But we must give simply the germ of each idea in particular concrete clothes; perception by the senses should precede the resulting pure abstraction. Thus should the abstract constantly alternate with the concrete; the empirical *periodically* precede the scientific on ever higher and more difficult planes of inquiry.

Moreover, only thus can due scope be given for the exhibition of those powerful varieties in intellect and character amongst the pupils upon the due development of which depends, obviously, the progress of the race. Here, finally, we note again the suggestiveness of our parallel for educationists.

Conclusion: Principle and Practice.

In conclusion, to avoid misapprehension respecting the principles advocated in this Study, it seems important to

emphasize the relation between educational principle and practice. All principles, I take it, represent but partial aspects of reality. Nothing, perhaps, is more fatal to progress and to success in teaching than the attitude of the doctrinaire—belief in the universal validity of any abstract principle or system of principles, and consequent insistent adherence to it in practice. Principles thus viewed and applied are life-killing mechanisms.

Any educational principle wisely used appears to reach its culminating point of effectiveness for the user at the moment when some fresh experience shows clearly and decisively that there are occasions when the principle must even be *reversed* in practice. The principle then appears actually to recede in importance as a consciously used tool, and to sink finally into subconsciousness as merely one of many similar factors influencing the judgement. There, rather than in consciousness, it is more likely to contribute to a sound balance of judgement that fits itself at each moment to the demands of new experience.

Not a finished piece of mechanism, then, but an ever-developing organism should become the character of a teacher's system of educational principles. The doctrinaire, to whom abstract principle is a tyrannical master, must develop into the realist to whom it is merely an excellent servant.

Neither life nor teaching, experience shows, can be conducted wisely on any system of abstract principles.

The truly great teacher, absorbed at first in the routine activity of gathering experience in any new sphere, deliberately, and even painfully, passes through the scientific stage wherein the principle rationalizing that experience rises into the daylight of full consciousness, and finally emerges as the creative and joyous artist in whom the principle has sunk again into the night of subconsciousness. And so with each new sphere of experience and the underlying principle, the process is re-traversed. The artist becomes again a hodman, the hodman a scientist, and the scientist becomes again the artist on a higher plane of power and activity.

In the power to realize this supreme stage of activity —the *artistic*—the teacher is born, not trained.

> 'Grau, teurer Freund, ist alle Theorie
> Und grün des Lebens goldener Baum.'

CHAPTER XXIV

APPENDIX OF MISCELLANEOUS POINTS

THERE are a number of miscellaneous points which belong properly to the work but were difficult to place suitably in the preceding chapters. They are therefore brought together here.

I

CHINESE NUMERALS.[1]

1	一
2	二
3	三
4	四
5	五
6	六
7	七
8	八
9	九
10	十
100	百
1,000	千
10,000	萬

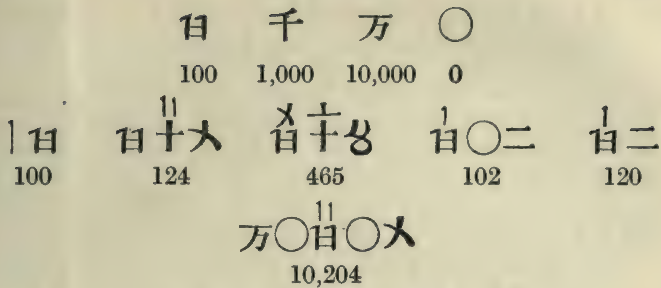
186,214 = 一十八萬六千二百一十四

Here the *additive* principle is used

CHINESE MERCHANT NUMERALS.

〡	〢	〣	〤	〥	〦	〧	〨	〩	十
1	2	3	4	5	6	7	8	9	10

[1] Cantor: *Geschichte der Mathematik*, vol. i.



Jewish or Hebrew Numerals.[1]

Formed from the letters of the Alphabet, as with the Greeks in Solon's time.

100 ק	21 כא	11 יא (10+1)	1 א
200 ר	22 כב (&c.)	12 יב (10+2)	2 ב
300 ש	30 ל	13 יג	3 ג
400 ת	31 לא (&c.)	14 יד	4 ד
500 ך (Final Kaph)	40 מ	15 טו (9+6) [2]	5 ה
600 ם (Final Mem)	50 נ	16 טז (9+7)	6 ו
700 ן (Final Nun)	60 ס	17 יז	7 ז
800 ף (Final Pé)	70 ע	18 יח	8 ח
900 ץ (Final Tsaddi)	80 פ	19 יט	9 ט
	90 צ	20 כ	10 י

e.g. 987 = זפץ on '*additive*' principle. For numbers

[1] For these I am indebted to one of my old pupils in Leeds, Mr. Phillipson.

[2] Observe the break here—due to יְהֹוָה (voh ho yay; voh) = Jehovah.

above 1,000 there was simple description in words. Observe that *Eastern* nations wrote from right to left.

II

Etymological Significance of the Names of the Numerals used by the American Indians of the Pueblo of Zuni.

One = 'taken up to start with' (i.e. the little finger on the left hand was taken up to count with).

Two = 'dropped or put down together with that which' (because the next finger was held up and put down with the little finger in counting *two*).

Three = 'partner equally itself which does' (because the third is the middle or dividing finger).

Four = 'all of the fingers all but done with' (because to indicate *four* all of the fingers except the thumb were held up and clasped down).

Five = 'the notched off or the cut off' (probably the whole hand held up with the thumb separated or notched away from the other fingers).

This minute analysis appears also in the higher figures, thus:

Ten = 'all of the fingers'.

A hundred = 'the fingers all of the fingers (done to)'.

A thousand = 'the fingers all of the fingers times (done to) all of the fingers'.

(Cf. threshing scores among the Romans, Roman numerals, pictographic Chinese numerals, tallies, knots, Peruvian quipus, &c.) F. H. Cushing (American anthropologist) *Manual Concepts: A Study of the Influence of Hand Usage on Culture Growth* (quoted in *Pedagogical Seminary*, p. 281, 1892).

Specimen of Use of Quinary (5) Scale amongst Inhabitants of New Caledonia. (Peacock's *History of Arithmetic*.)

Pārai	= one	Panim-gha	= six
Pā-roo	= two	Panim-roo	= seven
Par-ghen	= three	Panim-ghen	= eight
Par-bai	= four	Panim-bai	= nine
Pa-nim	= five	Paroonuk	= ten

III

Abstract of article on 'The Historical Development of Arithmetical Notation, &c., by L. L. Conant (*Ped. Seminary*, Vol. II, p. 149).

'Arithmetic as a science is the oldest of all the sciences. Arithmetic as an art is older yet, and its origin unquestionably dates back to the time when the human race was still sunk in deepest barbarism. The modern anthropologist has, upon the subject of number, made extended investigation among the savage races still in existence or but recently extinct, and has as yet failed to discover a single instance in which the number concept was lacking. It may be limited to the extremest degree, however, the entire number system of a language embracing but three or four words. The Veddas of Ceylon have but two distinct numerals, 'ekkamai' and 'dekkamai', one, two. Beyond this they count merely by the repetition of the word 'otameekai', signifying 'and one more', using this expression again and again (cf. Peacock). The Wiradmir of Australia count only to three. For four they use their native word for 'many', and for five 'very many'. The Puris, the Bushmen, and a few other tribes count only to two; the new Hollanders, the low forest tribes of Brazil, and others to three; the Abipones, the Carribus, the Galibi, &c., to four, and a numerous list might be given of the savage races which have five as the limit of their arithmetic. Others again count to ten, twenty, or a hundred, the latter being an exceedingly common limit among the more highly developed of the uncivilized races of the world. It should be noted, however, that the number system of a tribe is by no means an infallible index of their advancement in other directions. Certain of the most barbarous tribes of Africa, as the Yorubas, were rendered comparatively expert in numbers through their intercourse with slave-traders; and it is even asserted of this particular tribe that among them the saying 'You don't know nine times nine' is equivalent to 'You are a dunce'. On the other hand, the Peruvians, a highly civilized race, knew almost nothing of arithmetic as an art, and absolutely nothing of it as a science.

Briefly stated, the method of counting among savage tribes seems to be this: they make use of a small number

of numerical words, and anything beyond that is with them 'many'. Their scale may be one, many; one, two, many; &c., ... In practically all cases the assistance of the fingers is needed. *By means of these, counting is often done beyond the extent of their number vocabulary*, the total being indicated by holding up the proper number of fingers. If the number in question exceeds ten, the toes, or *the fingers of a second man*, are brought into requisition. (Cf. Peacock's statement of the *fundamental* method: sheep passing through a gate, and re-counting and verifying.) . . . From the simplest beginnings, arithmetical notation has arrived at its present state of perfection by passing through a number of well-defined stages or systems. This development, of course, has not been simultaneous all over the world.... Nor are they *all* to be met with in any *one* part of the world. The different systems are seven in number, and no people appears ever to have used *all*. Of these seven systems, four are now in common use in different parts of the world, and three of them are found in common use in all civilized countries.

1. The first principle of notation may be termed the natural principle. It consists merely in the repetition of the straight stroke, dot, or some corresponding symbol; e.g. 2 is II, 3 is III (three strokes), &c. Only among the least civilized is this system used in its purity. The ancient Egyptians did it for all numbers up to ten, but had a new symbol for ten. The common Roman system has it up to five, then a new symbol. The *ancient* Greeks and Romans, however, both indicated numbers by simple strokes as high as ten. The Aztecs carried this system as high as twenty, but they use a small circle in place of the straight strokes. All races have unquestionably made use of this principle at some period. The use of counters, or markers, as pebbles, shells, kernels of grain, &c., and the cutting of notches in a stick, Robinson Crusoe fashion, may be regarded as variations of this principle.

2. The repetition of strokes, the cutting of notches, the piling up of pebbles one by one, soon becomes confusing with large numbers. Hence, in the telling off of 5, 10, 20, or 100, a single stroke was made in a new place, or a pebble laid aside as the beginning of a new pile. Then this *new number* (5, 10, ...) is again told off by repetitions of the single stroke, and a second record made. This process gives

us the *additive principle*, so often met with in primitive systems of notation, e.g. Egyptians, ancient Greeks, Babylonians, Phoenicians, Palmyrenes, Hebrews, Romans, Aztecs, and, in fact, by *almost* all peoples that have emerged even into the rudest beginnings of civilization. An example is the so-called Roman notation, still used by the modern world for many purposes. This system is perhaps the best of its kind ever invented, for it uses independent symbols for 5, 50, and 500, &c., as well as for the powers of 10. The Egyptian, Phoenician, and other additive systems have only the latter symbols (10, 100, 1,000) and the Aztec system contains in all only 4 symbols, those for 1, 20, 400, 8,000.

3. A great advance on the additive system is to be found in the principle which, from the selection of its symbols, is called the *alphabetical* principle. In this we find the letters 1 to 9 represented by the first nine letters of the alphabet, the tens, from 10 to 90, by the next nine letters, the hundreds up to 900 by the next nine, if the alphabet contains a number of letters sufficient for the purpose. If not, a few symbols outside the alphabet proper are used. Thousands (and higher denominations) are usually represented by variations of the letters already employed. This system was in use with Syrians, Copts, Armenians, Ethiopians, and ancient Greeks, and was brought to its highest state of perfection by the last-named. In their system $a = 1$, $\beta = 2$, but $Z\beta = 12$; $\gamma = 3$, $\rho\kappa\gamma = 123$, &c. By this method any number may be represented by a number of *places* no greater than that required by our modern system, but the number of *symbols needed is much greater*. Thus $\eta = 8$, an entirely different symbol signifies 80, and still a different one 800, while we use but *one* symbol in writing 888. The Greeks used 3, writing it $\omega\pi\eta$. This system is far superior to any other used by the ancients, and is inferior only to the perfect system used by the modern world.

4. In numeration, we use, with all except small numbers, a system which, with a single exception, has never appeared in any mode of notation. We say 'one hundred', 'four thousand', &c., but when we write these numbers we do not write the one and then the hundred, or the four and then the thousand. A strictly analogous notation would require us to write five hundred thus, 5–100; eight thousand as 8–1,000,

&c.; uncouth as such an expression would seem, it is the system actually employed by the Chinese, and it was until very recent times universally employed by the Japanese. An idea of their system will be given by writing the number 26,438 thus: 2–10,000 6–1,000 4–100 3–10 8. However, since *single* symbols are used for 10, 100, 1,000, and higher multiples of 10, the actual Chinese method is really rather better than this illustration might suggest. (See p. 346, § I.) But it is unwieldy at best, and in Japan it is fast giving way to the Arabic system. A few examples of this multiplication system are found mixed with other systems, but they are rare, and the system itself can hardly be regarded as one of the steps in the development of notation from the piling of pebbles up to the 'Arabic' system.

5. An interesting variation of the alphabetical system (3), is the *marking* principle of notation. It is never used as the basis of a system, but is not infrequently employed in an auxiliary manner. In this system the same symbol, varied only by the addition of some distinguishing mark, may represent 1, 10, 100, &c. Thus, if a be the symbol for 1, we might let $\dot{a} = 10$, $\ddot{a} = 100$, &c.; if $u = 8$ then $\dot{u} = 80$, $\ddot{u} = 800$, &c. Then $108 = \ddot{a}u$, $808 = \ddot{u}u$, &c. The Greeks used this method with numbers from 1,000 to 10,000, and traces of the same system are found among Roman ruins. An exhumed tablet gives the number 1,180,600 thus: $\overline{\text{MCLXXX}}$DC, remembering that a bar multiplies by 1,000.

6. For the sixth and last stage at present reached, see pp. 364, 365 (i′) . . . (v′) of text.

IV

Examples of Old Algebraic Symbolism.

(Mainly from Tropfke, *Geschichte der Elementar-Mathematik.*)

Hieronimo Cardano (Italian mathematician) (*Practica Arithmetica Generalis*, 1539. *Artis Magnae sive de regulis Algebraicis Liber unus*, 1545).

	MODERN TRANSLATION.
5. p̃. ℞. m̃. 15.	
5. m̃. ℞. m̃. 15	$(5+\sqrt{-15})(5-\sqrt{-15})$
25 m̃m̃ 15. quod est 40.	$=25-(-15)=40.$
[p̃ = plus, m̃ = minus.]	

Johannes Buteo (*Logistica*, 1559, Lugduni).

	MODERN TRANSLATION.
$1A,\ \frac{1}{3}B,\ \frac{1}{3}C[\ 14$	$x+\frac{1}{3}y+\frac{1}{3}z=14$
$1B,\ \frac{1}{4}A,\ \frac{1}{4}C[\ 8$	$y+\frac{1}{4}x+\frac{1}{4}z=8$
$1C,\ \frac{1}{5}A,\ \frac{1}{5}B[\ 8$	$z+\frac{1}{5}x+\frac{1}{5}y=8$
$3A.\ 1B.\ 1C[\ 42$	$3x+y+z=42$
$1A.\ 4B.\ 1C[\ 32$	$x+4y+z=32$
$1A.\ 1B.\ 5C[\ 40$	$x+y+5z=40$
$3A,12B,\ 3C[\ 96$	$3x+12y+3z=96$
$3A,\ 1B,\ 1C[\ 42$	$3x+y+z=42$
$11B.\ 2C[\ 54$	$11y+2z=54$
$3A.\ 3B.\ 15C[120$	$3x+3y+15z=120$
$3A.\ 1B.\ 1C[\ 42$	$3x+y+z=42$
$2B.\ 14C[\ 78$	$2y+14z=78$
$22B.154C[858$	$22y+154z=858$
$22B.\ 4C[108$	$22y+4z=108$
$150C[750$	$150z=750$
$C[\ 5$	$z=5$

For example of Simon Stevin (1548–1620, Dutch mathematician), see pp. 259 (*L'arithmetique*, 1585).

Petrus Ramus (French mathematician, 1515–72).

$8q+9$	$8x^2+9$
$7q+4$	$7x^2+4$
$+32q+36$	$32x^2+36$
$56bq+63q$	$+56x^4+63x^2$
$56bq+95q+36$	$56x^4+95x^2+36$

Albert Girard, French mathematician, 1590?–1632? Also edited Stevin's works.

STEVIN'S INDEX-SYMBOL.	MODERN.
Soit 1 ③ esgale à 13 ① + 12.	$1x^3=13x+12.$
$(\frac{3}{2})$ 49.	$49^{\frac{2}{3}}.$

Table I below is an extract from *Les Œuvres Mathématiques de Simon Stevin, Augmentées par Albert Girard*, p. 169. (Published soon after 1634.) Table II is the modern form of the same. It is to be observed that the

notation is not Stevin's, but slightly more recent, the extract forming part of the matter added by his editor, Girard.

TABLE I

$$
\begin{array}{lcl}
\textbf{PRODUCTS.} & & \overbrace{\hspace{6em}\textbf{MULTIPLICATEURS.}\hspace{6em}} \\
\text{Binomes disioincts.} & \text{Binome disioinct.} & \text{Multinomes conjoincts.} \\
\left.\begin{array}{l}
Bq - Dq \\
Bc - Dc \\
Bqq - Dqq \\
Bcq - Dcq \\
Bcc - Dcc
\end{array}\right\} & B - D &
\begin{array}{l}
B + D. \\
Bq + BD + Dq. \\
Bc + BqD + BDq + Dc. \\
Bqq + BcD + BqDq + BDc + Dqq. \\
Bcq + BqqD + BcDq + BqDc + BDqq + Dcq.
\end{array} \\
\quad \&c. & & \qquad\qquad \&c.
\end{array}
$$

Constitutions de quelques triangles en nombres rationaux.

Subtendentes. Costez comprenans l'angle.

$Bq + Dq$ $\overbrace{Bq - Dq \quad BD2}$ quand l'angle est 90 degrez.

. . . .

$BD3 + Dq3 + Bq$. $BD2 + Bq \quad BD2 + Dq3$ quand l'angle est de 120 degrez.

. . . .

TABLE II

$$
\begin{array}{lcl}
\textbf{PRODUCT.} & & \overbrace{\hspace{5em}\textbf{FACTORS.}\hspace{5em}} \\
\left.\begin{array}{l}
B^2 - D^2 \\
B^3 - D^3 \\
B^4 - D^4 \\
B^5 - D^5 \\
B^6 - D^6
\end{array}\right\} & B - D &
\begin{array}{l}
B + D. \\
B^2 + BD + D^2. \\
B^3 + B^2D + BD^2 + D^3. \\
B^4 + B^3D + B^2D^2 + BD^3 + D^4. \\
B^5 + B^4D + B^3D^2 + B^2D^3 + BD^4 + D^5.
\end{array} \\
\quad \&c. & & \qquad\qquad \&c.
\end{array}
$$

Constitutions of some triangles in rational numbers.

Side subtending angle. Sides containing angle.

$B^2 + D^2$ $\overbrace{B^2 - D^2 \quad 2BD}$, when the angle is 90°.

. . . .

$3BD + 3D^2 + B^2$ $2BD + B^2 \quad 2BD + 3D^2$, when the angle is 120°.

. . . .

[Girard prefaces these results with the words, 'Let B be taken greater than D, and note that two letters joined together without the intervention of a point or other mark signifies the product of the same.']

Thomas Harriot, Oxford (1560–1621) (*Artis analyticae praxis*: London, 1631).

MODERN.

$$\left.\begin{array}{l} a+b \\ a+c \\ a-d \end{array}\right| = \begin{array}{l} aaa+baa+bca \\ \quad +caa-bda \\ \quad -daa-cda-bcd. \end{array} \qquad \begin{array}{l} (x+b)(x+c)(x-d) = \\ \quad x^3+bx^2+bcx \\ \quad\quad +cx^2-bdx \\ \quad\quad -dx^2-cdx-bcd. \end{array}$$

$$aaa-3\cdot bba = +2\cdot ccc \qquad x^3-3b^2x = 2c^3.$$

$$\overbrace{\sqrt{ccc}+\sqrt{cccccc-bbbbbb}}+\overbrace{\sqrt{ccc}-\sqrt{cccccc-bbbbbb}} = a.$$

MODERN: $\sqrt[3]{c^3+\sqrt{c^6-b^6}}+\sqrt[3]{c^3-\sqrt{c^6-b^6}} = x.$

William Oughtred (1574–1660, English country parson) (*Clavis mathematica*, 1631).

$$Aqq+4AcE+6AqEq+4AEc+Eqq.$$

MODERN: $a^4+4a^3b+6a^2b^2+4ab^3+b^4.$

Note that q = quadratus (square); c = cubus (cube).

$$Aqqcc+10AcccE+45AqccEq+120AqqcEc+\ldots.$$

MODERN: $a^{10}+10a^9b+45a^8b^2+120a^7b^3+\ldots.$

Isaac Newton (1642–1728: English mathematician and physicist, astronomer, Director of Royal Mint, President of Royal Society, &c.) (*Arithmetica universalis*, about 1685).

A DIVISION SUM.

$$\begin{array}{rl} yy-2ay+aa\,) & \underline{y^4-3\tfrac{1}{2}aayy+3a^3y-\tfrac{1}{2}a^4}\,(\,yy+2ay-\tfrac{1}{2}aa \\ & \underline{y^4-2ay^3+aayy} \\ & 0\ +2ay^3-4\tfrac{1}{2}aayy \\ & \quad \underline{+2ay^3-4aayy+2a^3y} \\ & \quad\quad 0\quad -\tfrac{1}{2}aayy+a^3y \\ & \quad\quad\quad \underline{-\tfrac{1}{2}aayy+a^3y-\tfrac{1}{2}a^4} \\ & \quad\quad\quad\quad 0 \quad\quad 0 \quad\quad 0 \end{array}$$

Note the use both of contracted and of uncontracted forms.

Gottfried Leibniz (1646–1716, German mathematician, philosopher, librarian, &c.). The symbols are almost entirely modern. Leibniz was perhaps the greatest constructor of symbolic notation—both in mathematics and logic—that has ever lived.

Jakob Bernoulli (1654–1705, Swiss mathematician) (*Acta Eruditorum*, 1690).

$$a+b+\frac{bb}{2a}+\frac{b_3}{2\,\mathrm{in}\,3aa}+\frac{b_4}{2\,\mathrm{in}\,3\,\mathrm{in}\,4a_3}+\frac{b_5}{2\,\mathrm{in}\,3\,\mathrm{in}\,4\,\mathrm{in}\,5a_4}+\ldots.$$

MODERN.

$$a+b+\frac{b^2}{2a}+\frac{b^3}{2\,.\,3a^2}+\frac{b^4}{2\,.\,3\,.\,4a^3}+\frac{b^5}{2\,.\,3\,.\,4\,.\,5a^4}+\ldots.$$

V

Note on Use of Fallacies.

To avoid both extremes—over-confidence in pure reasoning, and over-confidence in the use of instruments—and to bring into prominence the characteristic value of each avenue of knowledge and the mutual support they give each other when prudently used, we may with much profit as teachers follow the example of Euclid (in his lost (?) book of Fallacies) and the writers on Logic by utilizing 'fallacies'. A few are appended. Teachers will find it useful to make gradually a small collection for themselves, and without doubt original specimens will from time to time arise in class teaching which are worthy of preservation.

(i) Fallacy in reasoning, uncorrected by accurate drawing.

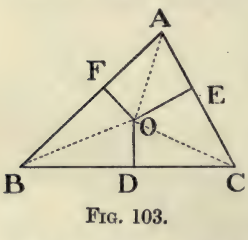


Fig. 103.

Select any triangle ABC. Bisect BC in D. Draw through D a perpendicular to BC. Draw through A a bisector of $\angle BAC$. If these lines do *not* meet, then they are parallel or coincident. In either case the bisector will be perpendicular to BC. $\therefore AB = AC$ (by congruence of triangles). If the bisector and perpendicular *do* meet, let them meet in O. Draw OE, OF perpendicular to AC, AB respectively. Join OB, OC. By Euclid I.

26 $\Delta AOF = \Delta AOE$. By Euclid I. 47 and I. 8 $\Delta BOF = \Delta COE$.

$\therefore AF + FB = AE + EC. \quad \therefore AB = AC.$

Similarly $AC = CB$. $\therefore \Delta ABC$, or *any triangle, is equilateral.*

The fallacy is readily seen either by actual construction with instruments or by paper-folding. It will be found that the perpendicular and bisector do not meet *within* the triangle. Consequently we must take the difference instead of the sum in the conclusion.

(ii) Fallacy in measurement, uncorrected by reasoning.

Take a square 8 inches in the side. [The subjoined figure is not drawn to scale; it is merely a sketch or type-figure.]

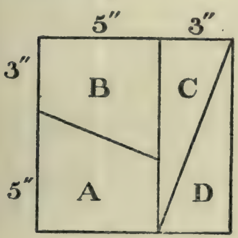


FIG. 104.

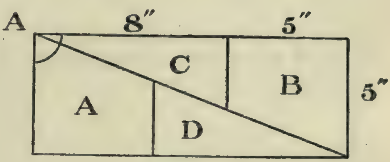


FIG. 105.

Divide it up as in the left-hand figure into four pieces $ABCD$. Cut these out. Re-arrange as in right-hand figure. Apparently a square containing 64 sq. inches has become a rectangle containing 65 sq. inches! Find the fallacy by *reasoning*. Use I. 47.

(iii) Fallacy in hasty generalization, uncorrected by test of application to a particular case.

Consider the formula $x^2 + x + 41$.

Let $x = 0$, its value is 41, which is a *prime*.
Let $x = 1$, „ 43, „ „
Let $x = 2$, „ 47, „ „
Let $x = 3$, „ 53, „ „

and so on. It appears always to be a prime for either positive or negative integers: many pupils after a few more

trials will draw this conclusion. *But*, for $x = 40$, or,—41, the function or formula equals 41×41, which, of course, is not a prime.

(iv) Combined fallacy of intuitive reasoning and experiment.

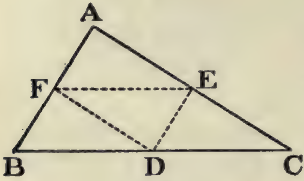


FIG. 106.

To fold *any* triangle into a tetrahedron.

Bisect the three sides at D, E, F respectively. Crease triangle along EF, FD, and DE. Turn the three corners about these three creases into coincidence at a vertex, thus making a solid—a tetrahedron in fact.

[The construction is not general: only acute-angled triangles can be treated thus. What happens if it is tried with right-angled triangles ?]

(v) Fallacy to show the final superiority of careful reasoning over even the most accurate kind of measurement humanly possible.

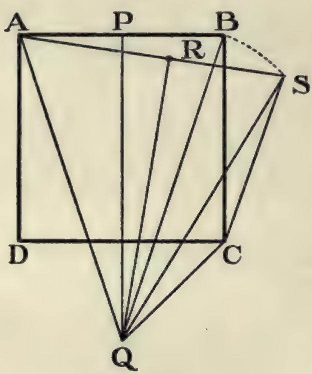


FIG. 107.

By careful reasoning Euclid I. 7 proves the universality of a certain truth.

To test this truth by measurement in certain particular cases. (Figure is *not* to scale.)

On any line AB describe a square $ABCD$. With centre C and radius CB describe a circle. Take any point S on this circle *near* B. Join CS, AS. Bisect AB perpendicularly by PQ, and AS perpendicularly by RQ. Join QB, QS, QC, and AQ. Then $AQ = QB$, and $AQ = QS$. $\therefore QB = QS$ and $CB = CS$, so that on the same base QC and on the same side of it we have two pairs of equal lines, &c. $\therefore$ Euclid I. 7 is false.

NOTE:—By *taking angle* BCS *small enough*, it is impossible, by instrumental drawing, i.e. empirically, to discover the mistake by appeal to measurement, however refined. Only careful reasoning can detect the fallacy in these cases where quantitative differences can be made as small as we please.

[The fallacy lies in the fact that the straight line SQ falls quite beyond SC, so that $\Delta\, SCB$ is contained *within* $\Delta\, SQB$.]

(vi) Take any square $ABCD$. Bisect the sides in E, F, G, H. Join each vertex to the mid-points of the 2 opposite sides, thus forming 8 lines which enclose an octagon, Test (1) by drawing and measurement, (2) by general reasoning, whether this octagon is a *regular* octagon or not, and state your degree of confidence in each method in this particular instance.

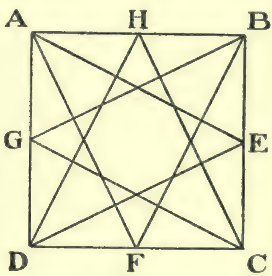


FIG. 108.

The octagon is equilateral, but not equiangular. Mere drawing and measurement—apart from reasoning—could not *prove* its rigorous equilaterality, though it might detect, with a reasonable measure of confidence, its want of equiangularity. Reasoning alone can establish the two facts for an ideally drawn figure. (An excellent discussion on the real nature of the two methods might follow here.)

(vii) For a much more difficult species of fallacy (involving the elements of the infinitesimal idea) see Chapter XXII, p. 318.

(viii) Discover the fallacy or fallacies in the following reasoning :—

(*a*) $x^2-1=(x-1)(x+1)$ for *all* values of x.

(*b*) $x^3-1=(x-1)(x^2+x+1)$,,

From (*a*) $\dfrac{x^2-1}{x-1}=x+1$,,

From (*b*) $\dfrac{x^3-1}{x-1}=x^2+x+1$,,

Now put $x = 1$, as the forms hold for all values of x.

$$\therefore\ \frac{1-1}{1-1} = 1+1 \quad \text{and} \quad \frac{1-1}{1-1} = 1+1+1;$$

$$\therefore\ \frac{0}{0} = 2 \quad \text{and} \quad \frac{0}{0} = 3;$$

$$\therefore\ 2 = 3; \qquad \therefore\ 2+1 = 3+1; \qquad \therefore\ 3 = 4,$$

and so on ; $\therefore$ *all numbers are equal.*

This fallacy affords an excellent opportunity of introducing 0 as an infinitesimal, and pointing out the possibilities of different orders of infinitesimals or small quantities. The elementary ideas of limits become thus gradually introduced long before the difficult symbolism representing them, and the rules for operating with them, are necessary. Frequent occasion should be taken to develop the *continuity* of algebraic and arithmetical ideas and magnitudes. Thus, in dealing with the meaning of x zero, it should be discussed, after dealing with $x^{1/n}$ (n, a positive whole number) in a simple way, purely arithmetically, without any reference to imaginaries, in close connexion with $x^{1/n}$. E.g. take $x = 2$ and n successively as 1, 2, 3, 4, 5 ... 10 ... (roughly approximating and using quite broad general reasoning, without descending to subtleties). Thus show that by taking n sufficiently large, $2^{\frac{1}{n}}$ may be brought as close as we please to 1, while $\frac{1}{n}$ approaches zero. Then take $x = 3, 100, 1{,}000, \ldots$. The truth then begins to dawn that, for finite values of x, however large, $x^{\frac{1}{n}}$ may be made as near as we please to 1 by taking n very large. In the limit (arithmetically speaking) we symbolize this by $x^0 = 1$.

Once the teacher realizes that perfect rigour of proof is an unattainable ideal even for the professional mathematician, he begins sensibly to aim at such a degree of rigour as his pupils may be reasonably expected to appreciate. The admission of this principle enables him constantly to introduce from the heights of mathematics fertilizing fundamental ideas into quite elementary teaching, and thus to avoid that disastrous break of continuity between school and college mathematics, and between academic and applied science, so characteristic of much of our

mathematical education. In resigning impossible rigour, the teacher must beware of falling into the other extreme, slipshod methods of reasoning. But to be forewarned is to be forearmed.

Again, the value of the limit-treatment of zero may be seen in dealing with the very elements of curve-tracing (say, the analytical geometry of the straight line); e.g. in deducing the case of *parallel* lines from the case of lines intersecting at a great distance from the origin.

(ix) Does the following construction trisect an angle? A young draughtsman offered me this as an exact solution by ruler and compasses alone. (Figure is not drawn to scale.)

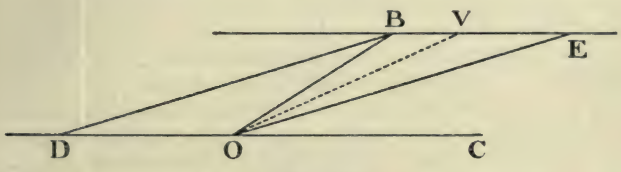


Fig. 109.

$\angle BOC$ is to be trisected. Take any point B on arm of angle. Through B draw parallel to remaining arm OC. Make $OD = OB$, and $BE = BD$. Then $\angle EOC = \frac{1}{3}$ of $\angle BOC$.

If $\angle BOC$ be very small, it is impossible by actual measurement to find any error in this construction. Theoretically, indeed, the construction, *if it could be carried out for such angles*, is correct for *infinitesimal* angles. The degree of correctness may be gauged from the fact that

for $\angle BOC = 30°$,
then $\angle EOC = 9° 56'$ (error about $\frac{1}{2}$ p.c.):
for $\angle BOC = 45°$,
then $\angle EOC = 14° 46'$ (error about $1\frac{1}{2}$ p.c.).

For angles $> 45°$ and $< 90°$, 'trisect', thus, the complementary angle, and subtract from 30°.

VI

Table Illustrating in Types the Main Stages of the Development of Fundamental Arithmetical and Algebraical Truths. (See also p. 252.)

The process is in the direction of increasingly wider

generalization, from early stages predominantly concrete to later stages predominantly abstract in form.

(i) Obscure mental stages. Self and not-self undistinguished.

(ii) Self and not-self growing distinct.

(iii) Perception of distinct elements in the body (fingers, toes, &c.).

(iv) This finger and that finger make two fingers (without language: direct perception, as with animals).

(v) Any finger and any other finger make two fingers. A repetition of this fundamental perception leads ultimately to creation of higher numbers, but still small numbers (three, four, five,...), but each number is for a long time attached indissolubly to discrimination of objects.

(vi) Thus: Two fingers and three fingers make five fingers.

(vii) Two cows and three dogs make five animals.

(viii) Two balls and three fingers make five things.

(ix) Two things and three things make five things.

(Logically (ix) precedes (viii): psychologically (viii) precedes (ix)—this inversion is *normal* in development.

(x) Further: Two and three make five. Here the number concept has become tolerably free, and is capable of a largely independent existence as an almost pure conception or idea, and can be readily re-applied to the interpretation of new sense-impressions, i.e. new experience.

Unless this ultimate freedom (never quite absolute and perfect) is attained by concepts in the various branches of education—freedom from the experience that gave them birth—the mind remains crippled and moves with difficulty over the vast field of life's experience.

At the same time there exists a certain final limitation to the degree of freedom the idea can attain, which ultimately depends upon the ability of the individual and the sphere of his experience.

How far this attainment of freedom depends on the use of language-symbols as literary expressions of the idea it is not easy to decide. Undoubtedly there are means of clear expression of the idea other than language: thus the artist (pictorial and plastic), the mechanician and the musician, have each powerful symbols of such expression, quite independently of the ordinary medium—spoken language.

We may group the above stages (between which numberless others could be interpolated by a subtle observer) together as, on the whole, *non-linguistic*, forming, say, division A. Division B (not necessarily always posterior to, but sometimes simultaneous with, division A, according to the nature of the predominantly powerful senses of the particular child) may then be taken as the rapid repetition of the above by the use of *spoken* symbols, i.e. ordinary language.

Subsequently comes the *written* form (B′) of the ordinary language. B and B′ we may call *rhetorical* divisions. Then come the groups of stages where written symbols peculiar to the art of counting are developed, say division C. Thus division C would comprise :—

(i) $||+|||=|||||$ (as a new grouping).

(ii) $2+3 = 5$.

(iii) $2a+3a=5a$ (a being finite, whole numbers).

(iv) $2a+3a=5a$ (a being finite, whole or fractional numbers).

(v) $2a+3a=5a$ (a being finite numbers, whole or fractional, positive or negative).

(vi) $2a+3a=5a$ (a being finite numbers, whole or fractional, $+$ve or $-$ve, imaginary or real).

(vii) $2a+3a=5a$ (a being transfinite numbers, with certain limitations); and so on, quite without limit.

(viii) $ma+na=(m+n)\,a$, with all preceding stages repeated on a higher degree of generality and abstractness.

(ix) $ma+na+pa+\ \dots\ =(m+n+p+\ \dots)a$, where, with certain conditions, there are perhaps an infinite number of elements.

(x) Ditto for :—

$$m(a+b\dots\text{ad}\,\infty)+n(a+b+\dots+\text{ad}\,\infty)+p(a+b+\dots\text{ad}\,\infty)$$
$$+\dots \qquad =(m+n+p+\dots\text{ad}\,\infty)\,(a+b+\ \dots\ \text{ad}\,\infty)\,;$$

with many conditions attached, the removal of any one or more of which leads to still higher generalities and abstractions.

Here, in C, we have not attempted to discriminate the logical and the psychological orders of development.

Any of these stages, relatively to a succeeding stage, is concrete; but relatively to a preceding stage it is abstract. The same truth may be both highly abstract and highly concrete according to the way in which it is regarded and the maturity of the individual employing it.

TABLE IN ILLUSTRATION OF SOME MAIN STAGES OF DEVELOPMENT OF NUMBER-SYMBOLS, AND THE PRINCIPLE OF *Place-Value* IN ARITHMETIC.

(i) *Distinct individual units;* pebbles, sticks.

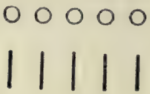


FIG. 110.

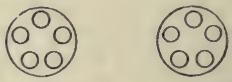


FIG. 111.

(ii) *Collective units.* Two groups of five pebbles represent ten.

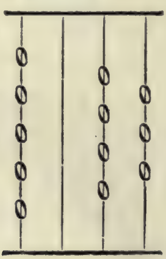
(iii) *Contracted collective units.* A single object (a shilling) represents twelve pennies.

(iv) *Pictures* of the above begin to be employed.

(v) *Contractions* of these pictures, leading ultimately to symbols whose origin is quite unrecognizable by the user, e.g. the modern symbols 1, 2, 3, ...

Along with the development of symbols for *recording* calculations goes a corresponding development of devices for making these calculations more rapidly and certainly, i.e. the evolution of pictorial symbols and calculating machinery proceed simultaneously. The two act and react upon each other until ultimately their final union produces the beautiful modern device—the principle of *place-value*. Thus :—

(i′) *Abacus* or Swan-pan, used by almost all large nations in one form or another.



represents 5043.

(ii′) *Graphic abacus*, with use of arabic ciphers and Roman headings.

M	C	X	I
5		4	3

(iii′) *Place-value, with embryonic zero* (indicated by •).

5 • 4 3

(iv′) *Place-value, with better-developed zero.*

5 △ 4 3   5 ⌂ 4 3

used in twelfth century by commercial houses: brought by Leonardo of Pisa from the Arabs.

(v′) *Modern place-value and fully developed zero.*

5043

not in full use till the fourteenth century.

Here again the general remarks in the preceding Table apply. Select, for example, Stages (iv), (iii), (ii), and (i). In (iii) the individual objects have to be imagined. Relatively to (ii), where the individual objects can be seen and handled, (iii) is distinctly abstract. But, relatively to, say, such a symbol as the number 12, Stage (iii) is as distinctly concrete. Even Stage (i) is already abstract in essence compared with, say, the actual living sheep which these pebbles symbolize and represent. In every case it is by a comparison of the *ideas* underlying, whether explicitly or implicitly expressed, that we must decide the degree of abstractness, and therefore of concreteness, of a symbol, and then it is only *relatively* so decided.

VII

Some Notes and Queries.

What proportion of our faith in the validity of arithmetical processes is due to the impress of authority in our youth, what to experience in the use of them, and what to an ultimately clear grasp of the nature of the fundamental principles and an understanding of the logic by which the processes are derivable from these principles?

Let each put the preceding question to himself concerning the statement,

$$32{,}798 \times 9{,}421 \times 74{,}692 \times 33{,}764{,}829 = 74{,}692 \times 32{,}798 \times 33{,}764{,}829 \times 9{,}421\,;$$

meaning by this:

If we multiply 32,798 by 9,421, the result by 74,692, and this result again by 33,764,829, we obtain the same final result as would be got by multiplying 74,692 by 32,798, this result by 33,764,829, and this again by 9,421.

The question, if properly understood and answered with insight, will perhaps lead to some surprising revelations of consciousness.

Imitation, both intelligent and mechanical (it is never *totally* either one or the other) plays a justifiable and highly important rôle in all psychic activities concerned with the acquisition of skill and knowledge. This is true even of mathematics in its severest disciplinary parts, and particularly applies to all highly symbolical processes. Observe how, in essentials of method, the Calculus of Variations is identical with the previously created Differential Calculus. Is it indeed [e.g. in observing the genesis of the Calculus of Variations by its greatest founder, Leonhard Euler (1744)] possible to distinguish creative thought from highly intelligent imitation, even in such work of veritable genius? The same piece of history, carefully examined, teaches again how science tends to pass from the inexact and un-rigorous towards the finally exact and rigorous, though never attaining the latter.

[Much to be recommended is the reprint and translation into German of the original works in Latin, on the Calculus of Variations, of the Bernoullis and Euler, in Ostwald's *Klassiker*, No. 46. The modern university student would profit much if introduced to this complex branch of mathematics by the study of the original essays of its founders. The gradual passage from easy geometrical intuition to the highly analytic stage reached by Lagrange (1762, 1770), Legendre (1786), and Jacobi (1837)—see No. 47 of Ostwald's *Klassiker*—is well adapted to the comprehension of youth at college.]

Exchequer: a superior court which had formerly to do

only with the revenue; so named from the *checkered* cloth (or draughtboard?) which formerly covered the table and on which the accounts were reckoned with counters (Old French eschequier = a chessboard). The Lords of Exchequer were called 'Barones ex Scaccario'. 'In reading old Middle-Age MSS. one notices—where there are money-accounts—*dots* at the margin. This is because the scribe, in adding, e.g. vii. d. and ix. d., made sixteen separate dots at the side and *then* counted these up. This is quite common in Middle-Age MSS.' Interesting and useful are such facts to the teacher of elementary mathematics. It is hoped that the few historical facts inserted here and there in this essay, and its whole tone, may induce teachers to obtain and study systematic histories of the science.

What is the ultimate proof or justification of any system of mathematical principles, methods, and rules? Is it not *self-consistency within the system* (i.e. the use of them never betrays a final contradiction)? Proof appears to be, ultimately, use. This principle of validity seems to be historically grounded, logically warranted, and forms the surest educational method. The expansion of mathematical analysis has ever continually demanded a widening of the system to meet this supreme need of self-consistency within the system.

The method devised in education for obtaining the required mechanical or automatic dexterity in any process —and particularly in mathematical education—is essentially identical with the process known in political economy as 'division of labour'. It consists in isolating all those operations which are identical (or almost identical) and performing these successively by themselves. Thus habit is rapidly formed by rapidity of succession, and time and energy are economized.

This is the educational rationale of 'sums' (drilling) in arithmetic, algebra, &c., and of 'scales' in music. It is justifiable, and even inevitable. But it requires much discretion in the teacher to judge wisely when the dexterity in manipulation has reached a sufficiently high degree of skill to form a sound foundation for the next advance in theory. The experienced teacher will know how to alternate

wisely between appeal to conscious intelligent effort and the automatic activity of a machine.

It is the effort required to transform the experience gathered subconsciously in mechanical routine into the conscious experience we name science that forms the great difficulty of all education, and predominantly is this the case in mathematical education. *But the danger of unduly straining the immature pupil in the too hasty passage from the routine activities to the scientific should be recognized by all teachers as truly great.* The modern reform movement must beware of this danger. Here, indeed, is where the insight, sympathy, and tact of the truest teacher is shown. Remember that whenever you appeal to reason, to logic, you are at once entering this difficult stage, and take care to return at once to the easy mechanical routine the moment you feel your pupil's attention and energy are being unduly strained.

What are the chief functions of rules, methods, tables, laws ? Are they these ?—

a. Bringing experience to a focus in consciousness.

β. For future application in the interpretation of new experience.

γ. To serve as a bond of association for memory.

δ. To economize thought, energy, and effort.

ϵ. To satisfy the unifying instinct (the '*Einheitstrieb*').

'*The Arabic Zero* is simply a *point*. As to the Arabic name of the zero, it is the word *sifr*, which means empty; *saharâ sifr* means empty space. This name describes the rôle of the zero and not its figure, and the Arabs have never applied this name to the other figures, as has been the case in Europe.' [See *British Almanac*, 1875 ; *The Arabic Numerals* (by De Morgan ?).]

On the Origin of Numerals. See *Letter to Nature*, vol. xiii, p. 47. Suggesting the genesis, Chinese → Hindus → Arabs → Europe. 'Numerals may be turned through 90° or 180°, or reversed, or inverted, without altering value, e.g. Arab "four" = , "inverted" gives = ; hence modern 4.'

Before passing to the more abstract form, wherein they

can stand alone (as $x+5x^2+7x^3$), the forms x, x^2, x^3 were originally conceived as so many units of length, area, and volume respectively. Thus they passed to their present abstracter form through the path of concrete geometry or mensuration.

It is where the concrete preliminary training of the senses is lacking that language, learned by rote, apes the appearance of knowledge in such deliverances, in examination papers, as this : 'A circle is a *straight* line, drawn at a certain distance from a certain point, and, being produced, the ends meet.' Or what shall we think of the training in observation of the youth who gravely tells you that 'on the surface of a ball an unlimited number of straight lines can be drawn'! There is some genuine observation—though the diction is somewhat lacking in precision—in the statement, written by a seventeen-year-old youth, that 'a plane is a surface which is smooth and level; not having any hills or lumps sticking up'.

The main line of mathematical development has repeatedly diverged at the same moment in several different directions, only afterwards to unite again. In such case we have, following our historical principle, a *choice* of main roads.

The Central Truths of Mensuration.

A. 1. Every rectangle (with commensurable sides) is decomposable into squares.
2. Every right-angled triangle is half a rectangle.
3. Every triangle is decomposable into two right-angled triangles.
4. Every polygon is decomposable into triangles.

B. 5. The areas of similar figures are as the squares on corresponding linear dimensions.

C. 6. The volumes of similar figures are as the cubes on corresponding linear dimensions.

For treatment of incommensurable magnitudes, see p. 201.

The teacher must develop the subject 'in relief', bringing into due perspective at each stage of the pupil's maturity the central and subordinate truths.

Statistics, with the simple elements of probability, will win for itself a place in the school of the future. Laplace made this prophecy over a century ago; it has still to be realized, but the first steps are already visible.

The Fluidity of Mathematical Symbols.

Mathematical symbols are to be temporarily regarded as rigid and fixed in meaning, but in reality are continually changing and actually fluid (cf. p. 372, § 3). But this change is so infinitely gradual and so wholly subconscious in general that we are not sensibly inconvenienced in our operations with symbols by this paradoxical fact. Indeed, it is actually owing to this strange truth that progress in mathematical science is possible at all. An excellent instance is the gradual evolution of algebra from arithmetic—a clear hint this for teachers.

VIII

Additional Note on Axioms and an Instructive Experiment. See also Chap. XXII.

The following is another illustration of an axiom valid universally for plane figures in space but false for solid figures in general. 'Magnitudes of the same shape and size are identical, and therefore superposable.' The falsity of this popular axiom when applied to right- and left-handed screws, pyramids, &c., gave enormous trouble to the Greek mathematicians, and ultimately led to the refined discovery of infinitesimals, though the creation of these into different 'orders' was reserved for a much later age. [Unless, indeed, as is highly probable, Archimedes thoroughly understood their nature. Thus he employed, to all appearance, in certain mechanical problems, what is essentially a characteristic, not merely of the differential calculus, but even of its still more subtle child, the calculus of variations.] Only actual experience of our space—of three dimensions—could show the falsity of the above axiom. Nevertheless, its sphere of validity is still wide, and, it is even conceivable that further experience (e.g. of space with higher dimensions than three) might restore its validity in a certain sense for space of three dimensions, while imposing corresponding limitations for space of higher dimensions.

My friend Mr. David Mair writes to me: 'I was a trifle incredulous about your experiment [Chapter XXII, p. 305]. But yesterday and to-day I have confirmed it.' He encloses the following note, which he kindly puts at my disposal:—

September 22, 1906.

'Three sheets of paper laid together, and three identical triangles cut out. Triangles labelled Lucy, Marjory, Baby. Asked Lucy if L and M were of same size. She fitted and said, "Yes." If L and B of same size. "Yes," by fitting. If she thought M and B of same size. "Yes." Why? Hesitation, then "Because they fit," and she whipped them together before I could prevent. "Besides, you cut them all out together." No trace of axiom that two things equal to a third are equal to each other.'

September 23, 1906.

'Three equal strips of paper given to Lucy. She measured first against second and third. Then I asked her if second and third were equal. She thought so, but when I asked why, she began to guess which was bigger.

'Then I gave her two other equal strips and a footrule. She measured each 9 in. Asked if they were equal she said, "Yes, because each is 9 in."

'She stood under mantelpiece, and, as I thought, just fitted. Suppose another girl came and stood under and just fitted, which is taller? "The other girl, because I don't quite fit; I am a centimetre too short." No trace of axiom.'

IX

Epitome of the Laws of Particular and Generalized Arithmetic and of Algebra.

(See Chapter XVII, pp. 252, 253, for application of first part of this.)

§ 1. Order of operations is from left to right, e.g.

$$(7+5)-3 = 7+5-3,$$

i.e. add 5 to 7 and subtract 3 from the result.

§ 2. 0, 1, 2, 3, 4 ... particular numbers } operands (upon
a, *b*, *c*, *d* ... number in general } which we operate).

+, −, ×, &c. { operators (symbols directing us to perform certain operations).

§ 3. *Brackets* are the symbol of the fertilizing idea of mathematical reasoning—*the principle of substitution*—the postulate that whatever is formally true for any letter in general is true for any combination of letters, provided these are juxtaposed in accordance with 'the rules of the game'. (Note subconscious elements entering here; cf. p. 370.)

§ 4. Relations between particular symbols, general symbols, and operators :—

$$\left.\begin{array}{r} a-a=0 \\ \dfrac{a}{a}=1 \\ 1+1=2 \\ 1+2=3 \\ \&c. \end{array}\right\}$$ (True for finite values only of a: zero values also excluded.)

§ 5. *Laws of Generalized Arithmetic.*

Table A.

The operators $+$, $-$; addition and subtraction.

I. Law of Association for addition.	$a+(b+c)=(a+b)+c.$
II. Law of Commutation for addition.	$a+b=b+a.$
III. Laws defining the relations between addition and subtraction ($+$ and $-$).	$(a+b)-b=a$ . . . (1) $(a-b)+b=a$ . . . (2)
IV. Laws of Association for combined additions and subtractions.	$a+(b-c)=(a+b)-c$ (1) $a-(b-c)=(a-b)+c$ (2) $a-(b+c)=(a-b)-c$ (3)
V. Laws of Commutation for combined additions and subtractions.	$(a+b)-c=(a-c)+b$ (1) $(a-b)-c=(a-c)-b$ (2)

Table B.

The operators $\times$ and $\div$; multiplication and division.

I. Law of Association for multiplication.	$a\times(b\times c)=(a\times b)\times c.$
II. Law of Commutation for multiplication.	$a\times b=b\times a.$

III. Laws defining the relations between multiplication and division ($\times$ and $\div$).

$$\begin{cases} \dfrac{(a\times b)}{b} = a, \\[2ex] \left(\dfrac{a}{b}\right)\times b = a. \end{cases}$$

IV. Laws of Association for combined multiplications and divisions.

$$\begin{cases} a\times\left(\dfrac{b}{c}\right) = \dfrac{(a\times b)}{c}, \\[2ex] \dfrac{a}{\left(\dfrac{b}{c}\right)} = \left(\dfrac{a}{b}\right)\times c, \\[2ex] \dfrac{a}{(b\times c)} = \dfrac{\left(\dfrac{a}{b}\right)}{c}. \end{cases}$$

V. Laws of Commutation for combined multiplications and divisions.

$$\begin{cases} \dfrac{(a\times b)}{c} = \left(\dfrac{a}{c}\right)\times b, \\[2ex] \dfrac{\left(\dfrac{a}{b}\right)}{c} = \dfrac{\left(\dfrac{a}{c}\right)}{b}. \end{cases}$$

§ 6. Note the perfect analogy between Tables A and B.

Table C.

§ 7. Connexion between Tables A and B.

VI. Laws of Distribution connecting $\times$ with $+$ and $-$. [The terms a, b of the multiplier are *distributed* amongst the terms c, d of the multiplicand.]

$$(a+b)\times c = a\times c + b\times c \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$
$$(a-b)\times c = a\times c - b\times c \quad . \quad . \quad . \quad . \quad . \qquad (2)$$
$$(a-b)\times(c-d) = a\times c - a\times d - b\times c + b\times d \qquad (3)$$

VII. Laws of Distribution connecting $\div$ with $+$ and $-$.

$$\begin{cases} \dfrac{(a+b)}{c} = \dfrac{a}{c} + \dfrac{b}{c} \quad . \qquad (1) \\[2ex] \dfrac{(a-b)}{c} = \dfrac{a}{c} - \dfrac{b}{c} \quad . \qquad (2) \end{cases}$$

§ 8. *Table D.*

(Laws of Indices.)

$$\text{VIII.} \quad \begin{cases} a^m \times a^n = a^{m+n} \quad . \quad . \quad . \qquad (1) \\[2ex] \dfrac{a^m}{a^n} = a^{m-n} \text{ (if } m>n) \quad . \quad . \qquad (2) \\[2ex] \qquad = \dfrac{1}{a^{n-m}} \text{ (if } n>m) \quad . \quad . \qquad (3) \end{cases}$$

IX. $(a^m)^n = a^{m \times n}$.

$$\text{X.} \quad \begin{cases} (a \times b)^m = a^m \times b^m \quad . \quad . \quad . \quad (1) \\ \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \end{cases}$$

§ 9. *Restrictions.*—All the operands (letters a, b, c, d, ..., m, n, ...) are pure signless numbers (integers or fractions); and where the index is fractional we have $a^{\frac{p}{q}} = \sqrt[q]{a^p}$, and $\sqrt[q]{a^q} = a$, where p and q are integers. All operations are strictly arithmetical; e.g. $3-7$ as a final result has no place or meaning in arithmetic (proper), but $+3-7 = -4$ means that an addition of 3 (to a preceding sum) followed by a subtraction of 7 is equivalent to a subtraction of 4; but the result must be combined with a number not less than 4, finally, to be arithmetically allowable (i.e. interpretable).

Statements of all these laws are possible in ordinary language; e.g. Table A, Laws (I), (II) may be read:—

(1) In any chain of additions we may group together ('*associate*') and change the order of ('*commute*') the operands in any way we please without altering the total value of the sum.

[The symbolic statement of the laws, in fact, should be preceded by their recognition and expression in ordinary language, after implicit use of them in operations with particular numbers in common arithmetic.] (See Chap. XVII, p. 253.)

Again:—Laws I, II, III, IV, V (Table A) may be thus stated:—

(2) In any chain of additions and subtractions we may associate and commute the letters in any way we please, provided we operate, in inserting or removing brackets, in accordance with the Laws of Signs, which laws are that, &c.

Precisely similar statements may be made for Table B (multiplications and divisions).

Finally note literary description of Table C, with Laws of Signs for $+$, $-$ with $\times$, $\div$.

The Fundamental Laws of Algebra—for higher forms.

§ 10. Algebra is essentially a calculus of symbols, with symbols and rules of operation, but with no meanings to those symbols over and above the points of meaning (sub-

consciously derived from arithmetical transformations), on the implicit observance of which depends the possibility of any formal transformation at all. Algebra is a generalization of generalized arithmetic. Arithmetic is, indeed, only one—though the most important—out of a possibly infinite number of equally *significant* algebras.

§ 11. (i) $(a+b)+c = a+(b+c)$.

(ii) $a+b = b+a$.

(iii) and (iv) $\begin{cases} (a+b)-b = a. \\ (a-b)+b = a. \end{cases}$

(v) $(a \times b) \times c = a \times (b \times c)$.

(vi) $a \times b = b \times a$.

(vii) and (viii) $\begin{cases} \dfrac{(a \times b)}{b} = a. \\ \left(\dfrac{a}{b}\right) \times b = a. \end{cases}$

(ix) $(a+b) \times c = a \times c + b \times c$.

(x) and (xi) $\begin{cases} a^1 = a. \\ a^m \times a^n = a^{m+n}. \end{cases}$

(xii) and (xiii) $\begin{cases} a-a = 0 & \text{(whatever finite value } a \text{ may have.)} \\ \dfrac{a}{a} = 1 & \text{(whatever finite value } a \text{ may have except } a = 0). \end{cases}$

§ 12. Also (for arithmetical applications): *Quantitatively* (connecting symbolical 0 with 0 as quantitative),

$$(a+x)-a = 0,$$

when x is infinitesimal.

§ 13. Resulting conceptions of positive, negative, and imaginary *numbers* and application to concrete magnitudes as positive and negative, with double function of operators $+$ and $-$.

$$+(+a),\ -(-a),\ -(+a),\ (-a) \times (-b),\ (+a) \times (-b), \text{ \&c.,}$$
$$\sqrt{-1}, \text{ \&c.}$$

§ 14. The only restriction on the operands is that division by 0 is illegitimate; also from the quantitative side, operands must not be infinite: thus, $\infty - \infty$ is not necessarily 0, nor $\dfrac{\infty}{\infty}$ necessarily 1.

§ 15. As *pure* algebra, no symbols have any significance (apart from the above laws to which they are subject) over and above those points of meaning on which depends the possibility of formal transformations.

§ 16. *The Application of the purely symbolical Science of Algebra.*

If the special symbols of any science or art whatever obey any set of the above formal laws of algebra, then *all* the theorems in algebra *deducible from that set of laws* are at once interpretable as truths in the before-mentioned science or art. We thus get an instance of a *significant* algebra.

[Examples of 'significant' algebras: Euclid, Book II, may be derived thus (see Encyclopaedia Britannica, 10th edition; article on Geometry by Henrici). *Theory of Physical Dimensions*: *Analytical Geometry*: *Differential Operators*: *Arithmetic itself*: *The Formal Laws of Thought* (see Boole) &c., &c.]

§ 17. The question as to which laws of algebra, as symbolical science, are *independent* and *fundamental* or *interdependent* and *consistent*, in the sense that the whole of algebra is rigorously deducible from them, appears to be as insoluble as the corresponding question in Euclidian geometry—probably also for the same reason, viz. the astounding subtlety by which intuitions enter subconsciously into all our reasonings. Also see the last paragraph of VII, p. 370.

X

The Treatment of Fractions.

All experienced teachers are aware of the difficulty of getting pupils to see clearly the logical basis of *fractional* operations in Arithmetic. The difficulty is due to the fact that there takes place an extension of the original meaning of the ideas, and of the scope of the operations, of multiplication and division. To expect the pupil to master the rationale of the process at the outset is unwise, and results in straining both teacher and pupil. Turning to our historical principle of racial and individual parallelism for guidance, we find the following five main features characteristic of the racial development of fractional operations:

(1) With a strong sense of faith, guided by a feeling of

analogy, and stimulated by the need for economizing calculations, our ancestors insensibly extended the rules of integral operation to operations with fractions. The dominating fact is that an integer itself may be formally expressed as a fraction, and that, too, in numberless ways; and, conversely, a fraction may turn out ultimately to be an integer. Here for the first time in Arithmetical operations the actual content begins to be masked in the form. We are entering the region of algebraic reasoning, without its special symbolism.

(2) This extension, whether tested by concrete application or abstract calculation, is found to lead to perfectly valid results. Faith in its validity is thereby strengthened.

(3) Simultaneously, the ideas of multiplication and of division also become insensibly extended, so that the ultimate test of the difference between multiplication and division lies not in actual numerical significance but in the particular way in which the operand (whatever may be its value) gets operated upon. This suggests the use of generalized arithmetical symbols (a, b, . . .), in a very simple way, to help, in teaching, towards the gradual elucidation of the whole process. The essence now is the operation, not the actual number operated upon; so that the use of a letter tends to obviate obscurity in thought, though apparently adding to the complexity.

(5) On looking back and reflecting upon the rationale of the use and experience of such extension of symbolism insensibly developed by the preceding generations, our ancestors, after much deliberation, succeeded in reducing the whole body of rules to a self-consistent, scientific system. (See also p. 367 on the ultimate justification of mathematical rules.)

Such also, in essence, has been shown by experience to be the most rapid and effective method of procedure in teaching, thus affording another illustration of the value of the historical principle itself. An analogy may make the point still more clear. In the early stage where we operate on integers with the four fundamental rules (addition, subtraction, multiplication, and division), Arithmetic is a comparatively simple hand-tool and the pupil a handicraftsman who sees, at each step, the actual meaning and

value of the operation performed. But when the fundamental operations are being extended to fractions, Arithmetic is fast becoming a more or less complicated machine, and the pupil (or mathematician) a mechanic who puts the material (numerical data) at one end into the machine, turns the handle (performs the indicated operations), and takes out the manufactured product (interprets it) at the other end. It is unwise to expect a full comprehension of the inner mechanism of the machine until some practical familiarity with its working has been gained, though an intelligent teacher will throughout be preparing the way for this full comprehension. A sound faith in the value of the machine and some wonder at its accuracy and power should precede any serious attempt to master the full rationale of its working. Both experience in teaching and the racial evolution of the index warrant a similar treatment of fractional indices in Algebra, and all other parts of mathematics where the original meaning of terms and the accompanying laws of operation receive radical extension in significance and scope.

Valuable as is the historical principle above used, it is inevitable that many incorrect applications of it will be made to teaching; doubtless this Study itself will ultimately be found to contain instances of such misapplication. This is largely owing to the fact that the story of mathematical development still contains many gaps and demands more thorough interpretation as an integral part of the historic movement of the whole experience of mankind in its manifold activities.

XI

Bibliography.

A list of works is appended designed to interest mathematical teachers in mathematical history and its application to education. The list is short—about twenty typical works have been selected—as the reader who desires to make a still more serious study of the subject will quickly extend his list by utilizing the bibliographies or references contained in the works here enumerated and in the additional works mentioned in the preceding chapters. Some

foreign journals on the teaching of mathematics have also been included. To teachers unacquainted with Italian or German the method is recommended of interesting themselves in these languages by tackling some foreign work on elementary mathematics.

Cajori, *A History of Mathematics* (Macmillan, 2nd edition, 1919).

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Chrystal, *Algebra* (contains many historical references and is already a classic work on its subject). (Black.)

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Mach, E., *The Science of Mechanics* (a critical and historical exposition of its principles. Translated from the German). Every teacher of Mechanics should study this work. (Chicago : The Open Court Publishing Co., 4th edition, 1919.)

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Einstein, A., *Relativity, The Special and the General Theory.* Authorized translation by R. W. Lawson. (Methuen & Co., 1920.)

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Cantor, Moritz, *Vorlesungen über Geschichte der Mathematik* (5 vols., Leipzig, Teubner). (The classic work on mathematical history.)

Tropfke, Johannes, *Geschichte der Elementar-Mathematik* (Leipzig : Verlag von Veit & Comp., 1902. 2 vols.). (A full and valuable work on the history of elementary mathematics.)

Ostwald's *Klassiker der exakten Wissenschaften* (Engelmann, Leipzig, Nos. 5, 14, 17, 19, 46, 47, &c.).

Pickel, *Die Geometrie der Volksschule* (Dresden: Bleyl und Kaemmerer, 8th edition, 1897). (An excellent textbook for primary school teaching.)

Schmidt, K., *Geschichte der Pädagogik* (4th edition, 4 vols.), by Dr. R. Lange. (Köthen: Paul Schettler.) (The classic work on educational history.)

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Weyl, H., *Raum, Zeit, Materie* (3rd edition, Berlin, Julius Springer, 1920). (A critical mathematico-historical presentment of Einstein's system of thought.)

Lavisse et Rambaud, *Histoire générale* (contains account of mathematical sciences by P. Tannery).

L'Enseignement mathématique, Revue internationale (bimonthly). (Paris: Carré et Naud.)

Enriques e Amaldi, *Nozioni di Geometria* (Bologna: Zanichelli). (A charming elementary work for secondary schools.)

,, Идея движенія въ современной геометріи и область ея примѣнимости въ курсѣ сребней школы.“

Докладъ читанный на 2-мъ Всероссійскомъ Съѣздѣ Преподавателей Математики 29. xii. 1913 г.

[The concept of movement in modern geometry and the field of its applicability in the courses of middle schools: a lecture read (by A. R. Kulischer) at the Second All-Russian Conference on Mathematical Education, December 29, 1913.] Attention is particularly directed to this paper, which deserves an English translation.

PART III

THE PAST, THE PRESENT, THE FUTURE

§ I. *Law of development in technique during Adolescence.*

DURING the conduct of an inquiry into the standards of accuracy in arithmetical work reached by boys and girls of all school ages amongst over seven thousand individual cases distributed amongst thirty schools (mainly secondary), I was strongly and continuously impressed with the distinct tendency to a decrease in accuracy at some variable point in the period of puberty. Further inquiries produced considerable evidence that a similar phenomenon obtains in other school subjects. This latter conclusion, indeed, would seem to follow from the former inasmuch as the function of number subconsciously or consciously underlies all our natural activities in their respective rhythms. Long experience verifies this deduction. Life has its '*Fourier-series*' the terms of which we shall doubtless gradually discover. This apparent law of development is of such importance to teachers that it is proposed to examine the matter with some care. The fact itself has been long more or less consciously or subconsciously familiar to many teachers whose experience has covered the major part of the pupils' school life, including the period of puberty. This general tendency to a diminution of accuracy is not always actually realized. There are many cases where the increase in accuracy continues steadily. There are still more numerous cases where the accuracy does not either diminish or increase, but remains for a time stationary: the individual appears to be temporarily exhausted and recruits himself in readiness for a further advance. Certain investigators image these stationary periods as *plateaus* of skill, and discover them at all periods of life for many kinds of activities. But the most numerous cases of all in school life—so numerous indeed as to form (it would appear from the present investigations) the greater number of the pupils—

are those where an actual diminution of accuracy during puberty is observable; and it is chiefly with this phase of the whole matter that we deal here.

The spiral of progressive mastery.

In developing gradual mastery in any field of activity the pupil appears to trace a spiral in which a series of three stages (or elements) continuously repeats itself. These three stages are—

A technical or routine stage.
A logical or scientific stage.
A creative or artistic stage.

The routine, automatic, mechanical element has its obvious bodily counterpart and is the expression of the deep subconsciousness of *instinct*. The logical, scientific element is the stage of *intelligence*, of the gradually awakening consciousness. The creative, artistic element is the transcendent phase of mind that, in analogy with instinct yet contrasted therewith, may be termed super-consciousness, or *intuition*. There is continuity throughout the three—instinct, intelligence and intuition—yet distinction, as with the colours in a spectrum.[1]

In the first, or routine stage, in which the fundamental technique appropriate to the new activity becomes automatic, its elements function at length with high mechanical accuracy and with least expenditure of nervous energy (psychic and corporeal). In a word, the tools become our own permanently; as, for example, the four operations of addition, subtraction, multiplication, and division in arithmetic: the symbols of shorthand: the pencil in

[1] For a more detailed and comprehensive study of the thesis, see the writer's *Janus and Vesta*, Chaps. XII and XIV (Chatto & Windus; 1916). May we venture to commend to the reflective teacher a patient comparison and contrast of this thesis of the composition of the mind's activity with the complementary thesis of the stages traversed in the evolution of 'proof' or 'evidence' in the study of geometry. [See Chaps. VIII, XI, XII, XXIII of the present work.] The essential interpenetrability and interdependence of ideas and their ever-startling Protean nature, must ever be borne in mind. All such analytical schemata are themselves but tools in the magic play of life's activities. [See also pp. 345 (reversal of principles), 367 (drill), and 376–8 (treatment of fractions).]

drawing: the brush in painting: the rules of grammar and syntax in composition: the saw, chisel, plane, and so forth, in carpentry: the metrical facility in iambics: the basal movements and actions in all games: scales in music: contour reading in geography: the footrule, vernier, compasses, balance, and so forth, in science: the scalpel in surgery: the spade and hoe, the scythe and pruning knife, and fork, in gardening: voice and gesture in oratory: and likewise with all the innumerable technical activities of humanity.

Here the mind is interested in the tools and in the adaptation of the body to the exploitation of those tools. This, we repeat, is the stage of *routine*, or *technique of automatic mechanism.*

In the second (logical) stage of development of power, the rationale, the essential logic, the scientific theory, based upon the preceding technique and yet also ever perfecting it, such is the aspect that now engages the interest and attention. *This logic is not in general linguistic.*

All principles are but deepening approaches to the ever-receding ideal of an interpretation of all reality. In this spirit, as practical teachers we may accept and apply the thesis here presented in a first crude way as equivalent to the old maxim: 'practice first and theory afterwards'. But as our experience grows and our methods become more finely adjusted to the subtle delicacies (ever to be freshly marvelled at with each new set of pupils) of mental growth, we see that both processes, readjusted automatism and readjusting logic, are ever proceeding simultaneously, but with alternative emphasis on each phase in obedience to that great law of rhythmic attention and energy, economy of mind and body. Thus even in the apparently simplest and crudest beginnings of infantile sense-experience there can be detected germs of logic, while to the loftiest and freest abstractions of the philosopher still adhere elements of automatism and solid chunks of sense—surprising as this may appear to many.

In fine, though, for practical purposes and analytical discussion of principles, we may justly separate the three stages enumerated, inasmuch as now one, now another rises temporarily into dominance, yet in reality the presence of all three elements may be detected. Thus a good kindergarten mistress will talk eloquently and truly of the highest element of all—the creative—as an indubitable quality of infants. Here we are rather concerned with the proportions of the three elements of activity and according to those proportions we have named the successive stages.

In the earlier or routine stage, the mind is primarily

directed to adjusting the living body with the various fine or large muscles to new tools or other additions to the environment. In the second stage, however, the emphasis of attention is transferred to the principles upon which the tools and technique work, with the object of adjusting these in turn to the body, and thus of forming them into still more perfect instruments by which we express our own particular gifts, and at which point the mind enters upon its third stage—creative or artistic. *Creative* it may be called, as therein the character reveals itself by its unique activity ; and *artistic* we may also think of it, inasmuch as the skill corresponding to the routine or technical activity primarily of the nature of a *craft* may rise thus into a true *art*, passing through and finally transcending the intelligent grasp of the science or logic inherent in a sound technique.

This second stage demands more concentrated will, more patient deliberation, sometimes even an apparent beating of thin air ; it is thus sharply contrasted with the concrete solidity of the technical or craft stage whose fruits can be so easily handled, and sooner or later are palpable in some evident way to the senses.

It is here that education brings its first touchstone that continuously selects and sorts out intellectual abilities. For though all have logical capacity, its degree varies between the widest possible limits.

In all things the highest form of skill, facile and adequate to its purpose, has traversed patiently a logical stage in which principles have been deliberately thought out, even though in the end these very principles become themselves transcended. To our highest achievement principles are as indispensable as the steel framework of a concrete structure ; or, still more pregnantly pictured, logic is as deeply imbedded in great art as the skeleton in the living body. We are all artists in our work ; but the scientific artist achieves on a higher plane than he who relies upon practice and intuition alone

In a word, the greater the underlying science the greater the resultant art as with a Plato, a Michael Angelo, a Dante, a Milton ; or with equally great men in other fields of human activity.

There is inherently no opposition between scientific power and artistic power, none between thought and action ; although excess of thought may paralyse action, while defect of thought may render action futile.

Finally emerges the third stage wherein the individual

having attained automatic dexterity and grasped the theory, principle, logic, or science underlying the operation (that is, working with rationalized skill), gradually reaches in each particular branch or subject of education his highest power, the power of creative or artistic activity. With the study of each new subject (or new branch of an old subject) the process repeats itself.[1]

The Distribution of Personal Energy.

Now inasmuch as the mind and body economize their respective and correlative types of energy by deliberate attention to one activity at a time, and find it therefore difficult, or even impossible, to attend to two processes equally and simultaneously, we should in general, from observation of our own nature and that of others, expect some diminution in the rapidity of growth of technical mastery when the logical stage is reached. This effect we should also expect, though in a diminishing degree, at all stages of life; but pre-eminently during youth when the mind and body first encounter the serious difficulties of both processes (automatic and rational) in the never-ended task of mutual adaptation of themselves to the world, and reciprocally of the equal readaptation of the world to themselves as a single personality. Now it may be as well to state that though this fact as a general truth appears to be subconsciously felt by most experienced teachers, it was not till statistical tests of school after school exhibited a drop in accuracy in the middle school and a rise in accuracy later that the truth of this general principle came fully manifested in that particular case of technique which we teachers know as arithmetical accuracy. From the results of these tests one might be tempted to conclude that there certainly existed considerable weakness in the particular schools tested. This result, however, conflicted with long experience of these particular schools; so that the question arose: 'A test of the middle forms certainly seems at first sight to take a fair and typical portion of a school as being equally removed from the beginning

[1] See *Janus and Vesta*; pp. 228–30.

and the end : is this supposition correct ?' Now reflection and experience alike suggest that the middle of any living process is, as a matter of fact, the most difficult of all to understand and judge, simply because the beginning and end are there inextricably intermingled. It is, to take an extreme but illuminating instance, as if we attempted to judge the nature of the caterpillar and the butterfly from a study, however deep, of the chrysalis that proceeds from the one and transforms into the other. More thorough statistical inquiry bore out the incorrectness of the former supposition and the broad justice of the latter view. Tests made on a more thorough and extended scale practically throughout the schools completely disposed of the original supposition. Every important fact militated against it: e. g. the drop showed itself in forms taken by teachers whose results in other forms were eminently satisfactory.

The law indeed revealed itself in such a high proportion of schools, in such varied circumstances, both for girls and for boys, that at length the fact forced itself upon one's notice that at some point during adolescence, after the onset of puberty, there is in general a diminution in the standard of arithmetical accuracy, followed normally by a recovery and even by an uplifting of the standard above that reached in pre-pubertal years. The importance of such a law or condition of intellectual growth is clearly so great and far-reaching that even the extensive and carefully worked out tests made in the present inquiry must not be taken as establishing more than its highly probable validity. It is eminently desirable that further and fuller investigations should be undertaken on the whole question by others. Even if its validity should be generally established, there remain such important questions as its limitations on the one hand and its possible wider fields of application on the other.

Out of thirteen boys' (secondary) schools tested for this law, twelve showed a drop in the middle school. In eleven of these twelve the lowest point of this drop fell between forms whose average ages lay between fourteen and fifteen, and of these twelve again the drop took place in six

schools in forms whose average age was approximately fifteen. Let it be noted that I speak of the lowest point of the drop, as a decline in the rate of growth of accuracy was frequently manifested before the ages mentioned. Again, out of eleven girls' (secondary) schools, eight showed the drop in the middle school (I use the term 'middle school' in a sense fairly well understood in secondary school organization); the remaining three did not exhibit any characteristic feature. But it should be added that, out of these three colourless tests, two schools were working under altogether abnormal conditions (reorganization of school work due to amalgamation of schools, or other equally disturbing conditions), and in the case of the third school there was not time to investigate with sufficient thoroughness certain perplexities that occurred. In no case examined (secondary or other) was there, after careful scrutiny, clear and substantial evidence of a violation of the law.

The drop in accuracy in the girls' schools took place also in general in forms whose average ages lay between fourteen and fifteen; but it is interesting to add that the average lay more nearly at fourteen and three-quarters—somewhat earlier than in the case of the boys. Again, in many instances there was a surprisingly quick recovery of accuracy.

It may finally be added that in an inquiry made in a class of twenty-three young women in a training college [average age nineteen] thirteen recalled definitely a period of decline in accuracy, in one occurring at the early age of nine, in others as late as fifteen, with an average of about thirteen. No great reliance, of course, will be placed on the particular ages recalled.

An educational law of this kind, suggested by and based upon statistical inquiries, however carefully conducted, requires something even more than further statistical investigations for establishment and support if it is to have legitimate influence and sway on the opinions of the teaching profession. We are bound in common sense to consult, as impartially as possible, the deliberately formed

opinions of experienced teachers. The law must also be subject to the test of observation of a few individuals continued over the whole period of life in question. This intensive study of the individual on the one hand, and, on the other, the garnered convictions of life-long teachers with experience over a great variety of forms and ages, form clearly two separate considerations, whose significance and importance are vital to the final establishment of any educational thesis based primarily upon statistical observations. The present inquiry has to some extent united these different tests for several years; and the evidence accumulated from all sides so far tends to strong general support of the law.

Yet to some teachers the law came at first as a distinct surprise and private tests (subsequently sent to me) were in some cases made by them of their own accord. These also corroborated the facts—frequently with unexpected strength.

Another test of a useful kind consisted in asking experienced teachers, previously unaware of the law, to state what variations, if any, had been found in their experience in the degree of accuracy shown by pupils at different ages. The results were somewhat remarkable. A small number thought there was a steady increase in accuracy from juniors through the middle school to the seniors. But nearly all those who had taught for many years all ranges of the age considered stated that their experience pointed to a distinct decline somewhere in the middle forms with a recovery of accuracy later; and in a few cases the average ages at which these variations occurred were hit at a guess with remarkable accuracy.[1]

Very numerous were the cases of pupils—especially boys—who seem to have 'gone all to pieces' in mid-school life but found themselves again later; unfortunately too there were cases where recovery had not taken place.

A very important consideration follows from the age at which the drop normally begins. For if (as appears to be the case from Dr. Ballard's[2]

[1] The great majority of experienced teachers in girls' 'Central' Elementary Schools where many pupils leave between fifteen and sixteen gave (mutatis mutandis) substantially the same reply.

[2] Dr. Ballard's tests dealt with *ages*, Mr. Burt's with *standards*. See

and Mr. Burt's experiments) arithmetical accuracy normally increases during the elementary[1] school age it would appear that, so far as arithmetical accuracy is concerned, in general pupils leaving the elementary school would be more valuable to employers than the secondary school pupil. One head master told me he knew a Railway Company that preferred boys leaving elementary schools at fourteen to boys leaving the secondary school at fifteen, as they find it easier to train them at this age to fixed habits necessary to the carrying on of the routine work. There is another important consideration. Under present secondary school conditions a large proportion of pupils leave at the age of minimum power of technique and maximum instability of character. In the foregoing considerations are powerful reasons for so organizing national work and education (elementary, technical, continuation, and secondary) that the attendance is maintained at school or the subsequent vocational years spent under wise guidance, till at least a reasonably large measure of stability of power and character is attained.

This seems the more desirable as a bulwark against the too frequent shattering waves of emotional stress that obtain during the mating period that follows.

Many were the reasons adduced for the existence of the law, which for shortness we may call *the curve of accuracy* (with its drop falling somewhere between the highest points corresponding roughly to the beginning and end of secondary school life proper). Some of these reasons are of considerable interest, and will be given in the next section.

From Childhood to Adolescence: the Passage from Imitation to Origination.

The famous modern French philosopher, Gabriel Tarde, attributed a rôle, permanent and high, to the existence of *imitation* in the march of culture and civilization. In childhood this rôle would seem to be at its maximum in the manipulative mastery, gradual and graceful, that the child obtains over its muscles[2] large and small (some of

P. B. Ballard's 'Norms of Performance in the Fundamental Processes of Arithmetic with Suggestions for their Improvement' (*Journal of Experimental Pedagogy and Training College Record:* December 1914 and March 1915).

[1] Which in London ends at fourteen.

[2] The once famous book *The Hand* by the great discoverer Sir Charles Bell, though written over two generations ago, still repays study by those teachers (especially mathematical) who can obtain access to it in libraries; it is pre-darwinian, but the anatomical *facts* are marshalled so lucidly and stimulatingly as to provide a most illuminating feast to teachers. In view of the renewed rapprochement after the long divorce between physiology and psychology, it is interesting and encouraging to recall that Bell's great nerve discoveries were the fruit of his

these latter inconceivably minute), a system of machinery surpassing all others known to man in united complexity, strength, and delicacy. Imitation of others is of such life-long importance that the nature of the surrounding models is of great moment. This is true pre-eminently also of the models (in arithmetic—methods and tables, neatness, figures, and other manipulations of technique) deliberately presented to the pupil by the teacher; for here the sub-conscious energy is reinforced strongly by deliberate and conscious effort on the part of the child, by emulation with other children, and by the normal desire to win the approval of the teacher. The teacher, indeed, in all ages of school and college education, stands more or less himself (or herself) as a model of conduct, intellect, and manners, to the pupils; and above all is this the case during the period of childhood. The importance of good models in the essential elements of arithmetical teaching (shape of figures, arrangement, methods, correctness, order, and so forth) is thus inexpressibly great in the laying of the foundations of the study. Childhood is the period when the technique of arithmetic and many other studies is being fixed and developed, for good or for ill, throughout life.

Then comes the life wave when childhood passes into the critical period of adolescence, whose initial stage is puberty. Puberty begins at very widely varying ages, so wide indeed as to be inconceivable to the layman and yet substantiated by reliable medical observation. The signs of its onset are not by any means always evident; but the experienced teacher will generally be aware of the fact and treat the young adolescent with some difference (naturally widely varying from case to case) in method and spirit from those suitable for the child as yet distant from this new and critical period of youth. We are not here dealing with medical or even hygienic aspects of education—although be it emphatically added that we may with assurance anticipate the full entry into the educational system of the medical or hygienic pedagogue. It is enough, therefore, to

continued observations on the correspondence between inner passion and outward muscular movement.

point out the importance of this new period in the life of the pupil with its momentous changes in body, soul, and spirit, in the deepening, enriching, widening, and uplifting of its emotions and intellect, of growth in ideals and senses, in will and in powers.

To all these psychological developments and re-orientings of the soul there would seem undoubtedly to correspond changes throughout the whole body itself; the latter indeed being the objective aspects of the inner and subjective phenomena, the two forming the complete reality, though the teacher in the main attends to the one, the doctor to the other. At this period emphasis passes normally from sense and sense-sprung intelligence of the world without to feeling and feeling-sprung intelligence of the world within, so that the mediating intelligence takes on a new colour. The intellect and judgement, based at first largely on sense, now repose rather on feeling. Feelings deepen into emotions, and these are interested pre-eminently in affection, truth, and beauty for their own inherent interest. Idealization in each of these three spiritual worlds of the soul rapidly develops. The soul is beginning to discover its own new and inner self. Its unique originality is becoming awakened into increasing activity. The inmost personality begins to emerge. Specific talent definitely manifests itself, sometimes in hobbies, sometimes in school subjects. New standards of value emerge.

Applying our analysis to the particular purpose in hand —mathematical education—we note that at this stage lively emotion and formal logic are not cold antipathies in the hands of a sympathetic and sound teacher, but natural allies: geometrical truth has its own compelling beauty. Yet we do well to remember, as ardent experts, that 'mathematical education and the education of mathematicians are two very different things' and not to be confused on pain and penalty of producing 'a mathematician the more but a human being the less'. The teacher himself (or herself) is now no longer considered in the same degree as the flawless model. Instead, a deliberate

selection is normally made of some attractive personality, whether among fellow pupils or teachers, parents or others. The new model becomes almost all-decisive to the future development for a considerable time. Normally this critical period (say roughly from twelve to sixteen with its culminating point at about fifteen) is traversed without serious disturbance to health. But not infrequently the strain, physiological regarded from the doctor's standpoint, psychical from the teacher's, is severe, and generous allowance must be made: understanding and sympathetic guidance must be given. By teachers unfamiliar with the conditions this cannot be done adequately.

The strong emergence of individuality brings with it a questioning attitude, a scientific trend. The logical instinct, the passion for symmetry and design, is awakened into life. In mathematics (arithmetic and geometry) the previous experimental and intuitional methods, though indispensable to increasing mastery at every stage of the study, take on a different emphasis in the school perspective and are rightly subordinated to scientific and logical developments: the systematizing and rationalizing age has patently begun.

We have said that puberty begins at widely varying ages. The Roman law, based on century-long experience, fixed twelve for the girl and fourteen for the boy. In our more northern climates the average ages at which childhood begins to pass into adolescence appear not to differ by more than one to two years from this figure. The rate of change in the new period may again vary widely: also the age at which relative instability (corporeal and mental) passes into relative stability of character and health, of interests and powers. The physiological expenditure of energy has its psychic counterpart. And so, as interest diminishes in the familiar technique, achievement in accuracy falls off normally. The energy is transferred to other fields of mental development, each with its physical or corporeal correlative. In mathematics, and in arithmetic in particular, the interest becomes predominantly focussed on principles, system, logic, and the whole of the

paraphernalia of rationale ; though here again we must not exaggerate : the stimulating reciprocity between theory and action, between science and its outward application must be developed throughout education, though the emphasis on internal and external will rightly vary at different times and periods.

Nevertheless there comes distinctly an age when the errant if charming fancy of childhood should gradually but decisively transmute into the disciplined imagination of youth, disciplined in this high sense that youth submits itself as willing disciple to the sovereign laws of reason. Life dreams at every age, but each age has its characteristic dream.

The pupil now begins to form more definite views and opinions ; he is building up a further stage of his *Weltanschauung*, his *World-orientation*. Further, between fourteen and fifteen experiments show that memory is at its least reliable stage.

For all these reasons, a drop in the curve of accuracy is, in the opinion of experienced teachers, to be expected during adolescence.[1] Many head masters and head mistresses were of opinion that new interests and extensions of the old formed also a weighty cause in particular of some inevitable diminution of interest in the old technique, and, therefore, would generally produce loss of accuracy in arithmetical operations. Carefully weighed, this particular cause will be found both consistent with and corroborative of the *general* cause or condition dominating the situation. We suggest here the question : 'On what grounds does each specific new study find entrance into the curriculum ?' Increasing breadth of studies forms the necessary field for the spontaneous growth of this special talent by each pupil. 'The importance of finding the work which makes an individual appeal', said an experienced mistress, 'cannot be overestimated.'

Consultation with teachers of other subjects (for example,

[1] In respect of boys an experienced master states that at this age it is the taller boys, rapidly growing, that 'fall off', but the 'little chaps' remain steady.

classics, drawing, history, manual work, music, modern languages, art, science, gymnastics, and so forth) brought out strong additional evidence of the substantial validity of this law of adolescence in all subjects of the secondary curriculum. Such wide extension, however, of the law beyond its original basis of discovery clearly requires far fuller investigation. Perhaps teachers in some of the above subjects will find an interest in testing the validity of the principle in the field of their own particular study.

The principle may perhaps be stated generally and provisionally thus:

In the continuous[1] *prosecution of a new activity the individual youth traverses successively three stages—technical, logical, and creative. In the second or logical stage there is normally a decline in the rate of progress of technique, but in the third stage (the creative) this decline is arrested and the final standard in technique reaches its highest point for that particular individual during his (or her) youth.*

Corollary: In childhood arithmetical accuracy in general steadily improves, but at some point after the onset of puberty it begins sensibly to decline, either absolutely, or at least in its rate of (growth)[2] *increase, and reaches its minimum during the period of adolescence: thereafter it rises and reaches its relatively highest standard as adolescence passes into maturity.*

Practical Consequences.

There remains the important and interesting question: 'What practical conclusions can we justly draw from this presumed law of mental and bodily growth?' To attempt a full answer would be premature and a final answer folly. We can offer only a few suggestions and record some opinions of teachers and others with whom we have carefully discussed the matter, and to whom we are greatly indebted for assistance in the inquiry.

First, it should be clearly stated that, out of a large

[1] Where there is discontinuity of any substantial kind the law requires substantial modification.

[2] This qualification is inserted owing to the perplexity introduced by the lack of any 'absolute' unit or standard of measurement.

number of teachers consulted, varying in temperament from the gentle skill that is imperturbably patient with stupidity and backwardness to the ruthless driving power of iron discipline, not a single one thought it would be wise to ignore such a natural law by a deliberate attempt with forceful methods to counteract the drop, and strive for a studied uniformity of technical achievement throughout school life.[1]

On the other hand, it was clearly held that in no wise should the law be interpreted as justifying deliberate slackening of effort on the part either of pupil or of teacher. Gathering up many conversations with individuals and useful little conferences with groups of teachers the following suggestions may be made upon this aspect of the matter:

(1) The presumed law emphasizes the immense importance of laying early and sound foundations in accuracy; of neat arrangement, clear and beautiful ciphering; of automatic mastery over the elementary tables; of plenty of drill both oral and written; of selecting the best methods and supplying the best models in the beginnings of all teaching.

(2) What possibility is there of deriving educational profit from some quality that, unlike accuracy or technique, *increases* during the very period in which accuracy or technique falls? For example, what of the rationalizing power, imagination, and idealization, the group or social instinct? Thus, early childhood is more isolated: adolescence more social. Investigation here also would be useful; and the many problems suggested are recommended to teachers here and there for careful consideration and inquiry.

(3) Special attention and sympathy should be given to exceptional cases where, for example, the disturbance to health or character, or both, is sub-normal or super-normal.

[2] (4) At all ages individuality reveals itself to the penetrant eye of sympathy. But puberty, the first stage of adolescence, is of such vital importance inasmuch as, in its psychic aspect of the gradual maturation of sex, it is then pre-eminently that the bent, the character, the genius, the personality begins clearly and unmistakably to reveal itself.

At this particular crisis of life there surely happens something to the

[1] We have spoken of the principle as both bodily and mental. It is an interesting physiological question as to what is the corporeal counterpart in nerves, muscles, &c., corresponding to the law of development of technique in the mental sphere. The scientist must assume in such an inquiry slight modifications in nervous organization as the correlatives of minute changes in the psychic aspect of life.

[2] See footnote p. 32 and *Janus and Vesta*, Chap. XIX; and especially pp. 229 and 230.

soul corresponding to the deep ploughing of the subsoil, a loosening of the relatively uniform and somewhat rigid character of childhood, with a view apparently to the selection and upspringing into daylight of some particular talent or genius which it is the honour and duty, perennial and inalienable, of the true teacher (ever surprised and delighted at the new plant heretofore hidden deeply in the rich soil) to help to bring to a healthy birth: and during adolescence to develop it to a vigorous shoot.

Or, again—to vary the imagery, for, man being a microcosm of great nature, all possible analogies have their particular and inherent justification and possess their special appeal to different minds—the machinery perfected in childhood and fitted thereto is temporarily thrown out of gear, does not run so smoothly (hence the fall in technical efficiency); there is an increase of just pride in expansion of new powers, yet some simultaneous loss of confidence in the exercise of the old; new courage, yet also new fears.

Or, again, from the vast ocean of subconsciousness, with its hereditary deeps, flows a strong tide into fuller and fuller consciousness, turbulent, muddy, and boundless at first, but at length growing clearer and shapelier, more distinctly directed to its goal, under wise guidance; but, if by mischance violently hemmed in, spreading out in shallow waters, and at length impotently and fruitlessly subsiding—perhaps for ever.

Great and noble then is the teacher's responsibility for the discovery, encouragement, and guidance of the pupil's particular talent, however humble.

A sympathetic reconsideration of the law of Adolescent Technique in the light of such points as the above may perhaps stimulate to useful applications that have so far escaped our own notice, and those not only in other than mathematical studies but even in the more restricted field of mathematics itself, to which we have here in the main confined ourselves. Observations of the older pupils in that *atmosphere of freedom* advocated for the younger by so many generations of great teachers (including Madame Montessori) might teach us much. A head master of eminence and experience some time ago, in the course of a private conversation, stated his belief that there was scarcely a single principle of education of which nowadays one could feel really certain, so much was everything in a state of flux. This, of course, was a somewhat exaggerated view of the position as uttered in private; but it well represents the prevalent atmosphere in things educational, and re-enforces the need for thorough and wide research in the educational field. The number of teachers now awake to the position is rapidly increasing; and yet an undoubted and great

harvest awaits a patient investigation by thoughtful teachers into the scientific principles of their own craft cultivating, as they do, so splendid an estate. The suggestion has often been made—and we desire to repeat it here, believing as we do in its great value—that the inauguration of something corresponding to what is known as a sabbatical year in American universities would give a great stimulus to educational research amongst teachers, if the grant of such a year were bound up with the condition that a substantial part should be devoted to refreshing and developing the teacher by travel combined with some inquiry into educational truths and investigation of foreign educational conditions.

§ II. *Specialists and Cosmology.*

There has been considerable specialization in schools of recent years. But here a danger has revealed itself in certain disadvantages that are attendant on specialization and specialistic organization of the school work (in sets or otherwise) to the detriment of those well-known advantages that flowed from the old organization by forms. This subject is too complex and wide to enter fully upon here. We have dealt with it in general terms elsewhere.[1] It is enough to point out that as there are three fields of nature in which scientific inquiry may be developed, namely, the mechanical sciences, the biological sciences, and the social sciences, so are there three branches of applied mathematics corresponding to these three fields. Hitherto the mathematical specialist has either confined himself to what is known as pure mathematics ; or at most in addition has generally embraced in his studies one of the branches pertaining to the field of the mechanical sciences (or mechanology), as statics and dynamics, more rarely physics (sound, light, heat, electricity, &c.) and still more rarely any of the other mechanical sciences, such as chemistry, crystallography, geology, astronomy (observational), &c. Mathematical crystallography has recently undergone striking developments that are making increasing demands upon all branches of modern mathematics (analytical and geometrical) and are influencing the fundamental concepts of physics. Many mathematical teachers would assuredly find it an attractive field of study, both in itself and as providing fresh sources of illustration for teaching.

Many years ago I mentioned the unfortunate general neglect of astronomy (both observational and elementary deductive) in our schools. Professor Nunn has recently emphasized this serious defect also, and made many valuable suggestions in connexion with it.[2] Until recently,

[1] See *Janus and Vesta*, Chap. X (*The New Humanism*) ; also *A New Chapter in the Science of Government* (Chatto & Windus).

[2] See T. P. Nunn's much to be recommended mathematical textbooks (Longmans).

most people probably thought the appearance of the heavens remained unchanged throughout the night; but the air-raids during the war focussed attention on the phases of the moon and incidentally on the diurnal rotation of the stars.

As we succeed in tracing the historical foundations of Hellenic geometry we find its primal philosophical elements deeply embedded in Egyptian systems of thought, alike religious [1] and cosmic, complex and grand. The great Hellenic pioneers of science were trained in Egyptian universities, and mathematics never wholly lost this theological and philosophical inspiration even after transformation by the Hellenic intellect, at once subtle and powerful, into a system of logical thought, compact and comprehensive.

Hence arose the great cosmic problems of the geometrical construction of the regular crystalline solids,[2] intervolved with the interpretation of the heavens. For their solution there was gradually forged through centuries of strenuous effort the wonderful chain of propositions that culminated in the grand synthetic work of the world-renowned 'Elementist', Euclid. This spirit it was—a union of religion, philosophy and cosmology—working within a definite concrete framework (the axioms, postulates, and definitions) towards a clearly conceived end (the construction of the regular solids) precisely as a great medieval architect designed and upreared a cathedral—that gave creative force, striking individuality to the elements and respective propositions, and artistic unity to the ordered march of the thought—a true world epic of the majesty of space. Far-reaching and immense as the common utility of 'Euclid' has been, still more have its endurance and influence owed to its grand artistry.

Once we attain a substantial realization of the history and dominant spirit of Hellenic geometry and contrast the conditions with those obtaining in modern times, less and less do we cease to wonder at the modern disintegration of Euclid, and more and more to understand that only in the steady and lofty light of a cosmic philosophy, still grander than the Hellenic, and adapted to the new conditions and needs of the whole human race (equally represented by women and men), can thinkers expect to succeed in a synthetic creation surpassing in symmetry, power, and breadth, that achieved by the ancient Hellenic genius, broad-based by wandering masculine merchant scholars on the whole wisdom and learning of the near and middle east. To such an ideal the woman must make her equal contribution.

With all its glories in individual branches the grand frame-work is still lacking to modern mathematics that alone can give synthetic unity to the foundations of science for the duration of the new coming Era, while preserving and even augmenting the sovereign rights of each separate branch and science within its own proper province—sovereign in the parts yet interdependent in the whole.

It is to be hoped—indeed there are substantial signs

[1] Typical in this aspect is the religious origin of the famous Delian problem (the duplication of the cubical altar to Apollo). Recall too, Plato's estimation of geometry as propaedeutic to philosophy.

[2] In this connexion note the inadequate position of 'solid geometry' in our schools.

of its advent—that in the schools of higher education will be found, in the future, mathematicians who have learnt to apply their science to the biological field, and others who have equally learnt to apply it to the social sphere. We refer particularly to economics, statistics, and finance. The Great War has shown a wide defect in education so far as comprehension of the financial bases of life and society is concerned. It is the mathematicians (both men and women) with a taste for economics who can perhaps in the future most hopefully contribute to a wider and truer grasp of what is implied in finance (political, commercial, and domestic, and their intimate interdependence), so that when their pupils pass directly into the walks of commerce and industry, or extend their studies further in the university, a sound foundation should have been laid in this aspect in the school. As for domestic finance, this should clearly receive more attention, and might become a popular field of mathematical study for those pupils who show little taste and capacity for the ordinary types of mathematical discipline. Women teachers may be confidently expected to achieve new educational successes in this field.

§ III. *Algebra and Mensuration.*

The advance upon the old Algebra has been distinct and great. Introduced as generalized arithmetic this branch has become vastly more comprehensible to the present generation of school pupils. The excessively long manipulations[1] by rote and many antiquated methods have largely disappeared. The application of algebra in union with geometry to trigonometry has also made a great advance. Practical surveying in the field has given interest, reality, and significance to the formulae. Text-books on the whole have improved. In a word, the re-orientation of the foundations of analysis (due, so far as teaching is concerned in this country, primarily to Peacock, Boole, De Morgan, and Chrystal) is influencing the schools

[1] Alike in arithmetic and algebra there is some tendency to the equally inadvisable *opposite* extreme—short sums.

with increasing strength and rendering the algebraic teaching more intelligible, systematic, practical, and scientific. In some secondary schools the elements of the differential and analytical calculus are also to be found.

There are weaknesses still; but steps have been firmly placed on the ladder of upward progress, and the existence of recently published systems of text-books providing stimulating suggestions to further progress is a happy sign that our mathematical leaders are wide awake in algebraical developments.

Touching smaller points—one is sorry not to see greater use of the Remainder Theorem, a proposition not easily matched for united usefulness, beauty, and simplicity; and more attention might well be given to the approximate solution of numerical equations of the second, third, and higher degrees. [See p. 219.]

In the upper forms the subject of probability and its simpler applications to vital and other statistics deserve more attention, with some introduction to the philosophical groundwork of 'choice and chance'.[1]

According to their staff and other opportunities schools might, with advantage to themselves and to the broad public welfare, even within the bounds of a single subject such as mathematics, after laying a broad and sound foundation, specialize on one or more branches in harmony with the modern growth of science. Such a movement the universities could both encourage in various ways and draw profit from it. *Provided the integrity of culture as an artistic unity of its several branches is simultaneously fostered and maintained in the whole school atmosphere, specialization is not only safe but a necessity for advancing movement, for thoroughness of scholarship and for provision of adequate facilities for varying talents. Moreover, only by the evolution of an elastic variety of curricula can be avoided the grave difficulties of one increasingly congested and common curriculum.*

With respect to mensuration, evidence is ample that considerable skill in handling and applying formulae is not sufficiently accompanied with thorough grasp of the great central truths of elementary (Euclidian) mensuration set out in clear perspective. See A, B, C, p. 369.

A rigorous proof on modern lines of the two last theorems (B. 5;

[1] A lucid presentation of important educational applications of the simpler elements of statistical science is provided in Dr. P. B. Ballard's *Mental Tests* (Hodder & Stoughton, 1919).

C. 6; p. 369) would, of course, be outside the range of school mathematics. But an appeal to simple intuitions, based on graphical illustration and careful experiment, suffices to establish all these truths, in ordinary cases, sufficiently rigorously for simple practical applications, while providing likewise a valuable discipline in geometry.

Let us recall that the last proposition of the twelfth book of Euclid establishes the theorem on which C. 6 above is founded. But this book of Euclid is not now studied in the schools, so that the above system of truths, though implicit in the formulae of mensuration, seldom culminates consciously and deliberately, as it should do, in the crowning proposition C. 6.

It was this crowning achievement that made into an epic the first twelve books of Euclid; and, in passing, let us note the probable subtle influence of the famous *twelve* in books of great poetry on the highly cultured mathematician of the world-renowned literary centre and university of Alexandria, enduring its thousand years.

It is just the lack of such a crowning problem that deprives our mathematical studies of powerful aesthetic unity and tends to reduce them so often to things of shreds and patches.

The following *viva voce* questions (or equivalents) I have put repeatedly to pupils in the upper forms of secondary schools or to students (men and women) in training colleges who have passed through the sixth form of a secondary school and in many cases have obtained a science degree:

Two potatoes are of the same shape and the same stuff: the circumference of one, taken with a piece of string, is twice the corresponding circumference of the other. Compare their weights.

Chang is a giant and Ching is a dwarf. They are exactly alike in every respect save size. If the thumb of Chang is three times as long as the thumb of Ching, how many times is the giant heavier than the dwarf?

A correct answer—even an approximate answer—has rarely been obtained even from such picked pupils. The first correct reply was given by a student who attributed his success (justly or not) to the fact that he used to sell eggs by weight in his father's business. Here, we submit, is food for fruitful educational thought.

§ IV. *Comparative Algebra.*

We pass to a still wider standpoint. Despite all the improvements in the teaching of Algebra and the truly admirable unifying efforts made by the late Professor Chrystal and his predecessors, and continued in the same broad spirit by the stimulating and suggestive school-mathematical works of Professor Nunn [1] and other scholars in quite recent times, we are afraid that a certain criticism still holds good that has more than once been made by thoughtful mathematicians. We may perhaps put it thus. It has been well said [2] that if it were not for the study of plane geometry in our schools (and even there it is

[1] Longmans, Green & Co. [2] By Professor E. V. Huntington.

becoming increasingly uncertain) it is doubtful whether our pupils would ever derive from the study of arithmetic and algebra alone any clear notion of what is meant by a mathematical demonstration, and, we might add, a system of sound logical thought. Algebra—we speak of its later stages—indeed is unfortunately still, as presented in the schools, essentially a science without real continuity, logical consecutiveness, and clear perspective. It is but a more or less compacted body of miscellaneous facts and rules. It never becomes a developed science. The pupil is constantly finding himself in perplexity through the unwarrantable extension of the meanings of terms due to the fact that definitions, alike too narrow and also too rigid, are attempted at the start. The position in algebra, and, of course, also in arithmetic—though to a less degree—is to a large extent the opposite to what obtained in Euclid. In Euclid we have had too abstractly rigorous a conception and system which require adapting to the immature intelligence, while in Algebra we have a miscellany of facts and rules that require some bracing-up into a more consecutive system in order to be worthy of the growing logical intelligence of our older pupils. It is as if in geometry we treated the logical faculty of order, theme, and system as perfected in our young pupils, while in algebra we regarded it with indifference if not with some measure of contempt. Now such a reorientation in algebra is not possible until our mathematical teachers in the schools become familiar with the ordinary algebra as one—though a very important one because it is a generalization of common arithmetic—of many Algebras. There is, indeed, a science of *Comparative Algebra* as well as a science of *Comparative Geometry* and *Comparative Mechanics*, the latter of which we propose to consider in their educational bearings in the following sections.

But here let us make well a distinction and avoid a confusion. It is not suggested that children should be taught Comparative Algebra or Comparative Geometry or Comparative Mechanics. But it is suggested that mathematical teachers should be familiar with the elements of

these great, powerful, and beautiful modern sciences; above all, the conditions would then be present for the gradual achievement of an ultimate unified presentment of mathematics from kindergarten to university, with its several grades each adapted to the intellectual maturity of the pupil in its large outlines with such branchings and applications as broad occupational needs may demand. The steering of the school course would then undergo considerable modifications and additions for the better. It is, we submit, a misleading simplification of facts to think of, and to teach the young mind on the assumption that it proceeds merely from simple to complex. Say relative simplicity to relative complexity, and we hit the mark more closely.[1] Better still, may we not say that the child-mind is a highly complex embryo yet possessing a strong instinctive unity? To analyse out and develop to maturity the valuable elements of this mind complexity while ever preserving the unity: that surely is one of the supreme achievements of the genuine teacher. Uniting this consideration with the foregoing may we not justly hold that a mathematical teacher, equipped as suggested, will be in a position, far superior to what he is at present, to present to the growing mind a mathematical science, sound as far as it goes, and forming a whole that both satisfies the instinct of unity and forms an incomparable instrument for the measurement and understanding of natural phenomena? In harmony with our general historical principle of parallelism in racial and individual development (see Index, s.v. Parallelism), the geometry of Euclidean space (as at once convenient, familiar, traditional, and historically justified) can be taught in the spirit and light of Comparative Geometry (or *General Geometry* as eminent French mathematicians call it); similarly the mechanics of the Galileo-Newtonian mechanics can be presented in the spirit and lights of Comparative Mechanics,[2] and, similarly, the algebra of generalized arithmetic (on identical warrants) can be taught in the spirit and light of

[1] See Chaps. V, VI, X, XX, XXI, XXII of present work.

[2] We refer, of course, to the modern developments of Einstein and others.

Comparative Algebra (or 'Universal Algebra' as it is often called): and both may be made contributory to the retention of a generous open-mindedness to new ideas and ideals on the part of our pupils, a quality of the soul that becomes a possession priceless to maturity and advancing age and yet one that is too commonly lost even in youth owing in large part to the excessive rigidity of the framework of education.

A movement is slowly gaining ground similar to the one here advocated in all branches of study. Thus we find it in linguistic study [1] necessitating on the part of language teachers some familiarity with the science of Comparative language. It has been critically objected to this that only teachers of genius can utilize such advanced and generalized ideas and ideals in the schools. But experience has shown that the process of evolution does not depend upon such precarious and fortuitous conditions. Genius persuades and leads, creates and inspires. Talent then takes up the work, systematizes and thematizes the initiatory work of genius in text-book and treatise, in seminar and lecture. Finally, professional ability applies (at times with genius, frequently with talent, generally at least with common sense) the erudite labours of talent to the daily task of the ordinary classroom. In a word, the grand hierarchy of human capacities co-operatively evolves each new advance, in which each degree of that hierarchy is necessary to the others.

In all spheres of life the process is substantially similar. Here perhaps we reach our most significant conclusion on the whole matter: for the same kind of difficulty really exists in educational science itself. It is split up into too many disparate and disconnected branches; nor do we realize yet the infinite significance of the various great periods themselves of human life forming in all a great unity. We have made some attempt to lay bare the outlines of a unifying law obtaining in the period of adolescence and to some extent applicable to all periods of life; but it is clear to all thoughtful teachers that the root problems, some of which we have indicated, confronting educational science and practice, are bound up with the creation and discovery of some slowly evolving unification of world-thought based upon the great periods themselves of the life of man.[2]

§ V. *Geometry.*

Since the dethronement of Euclid the geometrical school world has been in a state of unstable equilibrium [3] throughout the globe.

[1] See the admirable notes on Method in *The Times Educational Supplement*, February 1, 1917 (Language).

[2] See also the writer's 'Map of Life' in F. Watts's *Education as Self-realization and social service* [New Humanist Series: University of London Press, 1919].

[3] For analysis of the deeper grounds of this, see Chap. XI, *Janus and Vesta* (op. cit).

Text-books appear yearly by the score, most of them doubtless hoping to achieve the glory of substituting themselves for the old Euclidean system. It has been the writer's duty to study the more important of these (including many foreign ones) with considerable care. They vary greatly in quality and scope, in treatment and immediate object. Some of them reach a high standard of attainment.[1] Yet it is certain that Euclid has not yet found a single successful rival whether we judge by clearness and brevity or by beauty and symmetry, or by system and method, or (regarding the matter from another standpoint) by general mathematical approval. Despite invaluable contributions to the problem of school geometry by mathematicians of European eminence in modern times—practically beginning with Clairaut (1741) and Legendre (1794) in France—no general agreement, it seems, has yet been reached in any country in the world—not to mention a world-wide agreement. Though in almost every modern civilized state immense and admirable efforts on the part of the intellectuals have been devoted to this question, the ideal substitute for Euclid has yet to be found.

Perhaps the problem—as indeed a few eminent reformers are beginning to think—must be tackled from a broader standpoint. So closely interdependent are now all branches of mathematics that, as so often happens in mathematics itself, it may be easier to solve the larger problem before obtaining a solution of the smaller. Are these reformers right in holding that the problem is not that of a substitute for Euclid and therefore concerned with geometry only, but rather a question of a unified elementary presentment of mathematics as a whole (including both time and space, conceptions and experiences), in which, while the separate branches, arithmetical, algebraical, geometrical, mechanical, and so forth, would receive due specific attention as they arise out of the whole, yet the dominant aspect of mathematical education would be that presented by a veritable unity of constructive thought and practical application, constantly derived from and verified by the vital experience of the pupil? Such a solution would be in harmony with the fertile principle connecting the evolution of knowledge in the individual with its evolution by the race. It is not the precise form with which we are here concerned so much as the spirit permeating our knowledge throughout education.

[1] E. g. an excellent attempt, based on elementary notions of rotation, translation, and folding, to co-ordinate geometry, trigonometry, mensuration and the methods of co-ordinate geometry is made in *A School Course in Geometry* by W. J. Dobbs, M.A. (Longmans, Green & Co.).

The racial evolution in mathematics would seem to have been from an embryonic yet, in its degree, rationalized unity towards a subsequent multiciplicity of specialisms. But the science must, both in race and in individual, periodically recover its primal unity of spirit under the penalty of degenerating into isolated, inimical, and excessive specialisms.

The modern scholastic forms of geometry in substitution for Euclid have lost much and gained much. Among the qualities lost or greatly diminished are these: the real and many advantages alike for teaching and examining (too obvious to enumerate) that belong to a settled and universally accepted order of theorems; a thoroughly-tested pregnancy, brevity, beauty, and lucidity of style—producing a model so impressive and even sacrosanct that the propositions were for several generations of pupils learnt by rote, with but a glimmer of understanding into their significance by the vast majority of the pupils: and finally there was the uniquely intellectual stimulant Euclid provided for the pupil with genuine mathematical talent.

The present state must not, however, be exaggerated into one of predominant loss. The subject is vastly more luring to the pupils, if too often the mind remains in some obscurity—as to order and reasoning. We have to bear several considerations in mind. Logical rigour is never absolute.[1] That degree of rigour is best that is fitted to the degree of maturity of the understanding to which it is to appeal.[2] This latter is a vital educational principle in mathematics which is now formally established by the leaders of modern thought on the foundations of the science, and abundantly supported by the experience of thoughtful and practised teachers.

Equally important is another consideration. The critic who confines himself in the main to the examination of written papers and finding therein loose statements and apparent deficiencies in logic is apt to assume far too hastily that the teaching has been ineffective and the time devoted to geometry largely wasted. Such mournful conclusions are natural where the experience is mainly limited to the examination paper. But where a substantial part is taken in the teaching or continuous observation is made of the actual work in the school itself, and the practical fruits of the teaching brought to a different kind of test, then

[1] This is disputed by many great authorities. The question would seem to turn upon the significance attributed to the concept of 'time'. It may be that there *subsists* a great 'invariant' to whose realisation human approach is asymptotic, and which plays in philosophy a rôle like unto that of gold in finance. Such a conception may conciliate the two opposing views upon proof. [See also p. 141, footnote and Chap. XXII.]

[2] On this see Chap. V (especially pp. 98–101) Professor Whitehead, *The Organization of Thought, Educational and Scientific* (London: Williams & Norgate, 1917). This collection of essays by a mathematical thinker of European reputation may be warmly commended to all teachers for its width of stimulating thought; though the present writer ventures to differ as to the validity of some of the author's philosophical principles (see later sections of this report). On this important point of rigour of proof see Chaps. VIII, XI, and XXII of present work, and Chap. XI of *Janus and Vesta* (op. cit.).

the conclusions reached are very different. The critic now begins to recognize that the growing mind must in general traverse what we may call—it is a phrase frequently heard by us in the schools—*a working knowledge* of any subject (here geometry), before a logical, precise, and systematic structure can be built up by and in the mind. Such a 'working knowledge' does not enable the pupil to enunciate his processes, statements, and reasons with lucidity or brevity, or indeed with any but a loose form of logical consecutiveness; yet it does stand what every practical teacher knows to be the real and final test of grip of the significance of a truth, namely, power to use it in practical application with certainty, confidence, and accuracy of result. It is a case again, though of course not in anything like so extreme a form, of the tongue-tied practical man. He can do the thing but he cannot clearly tell you how it is done. This power and test of 'doing' are vital to all sound education. We repeat that this 'working knowledge' is a necessary and valuable stage; though education has not done its best unless it becomes transcended in mathematical education by a further stage of achievement in which the power has been developed to enunciate, from amongst a given system of propositions, clearly, briefly, and consecutively, those in particular upon which the decision has been taken.

We deprecate, however, the discontinuity in logical treatment that now far too frequently prevails. It is unnecessary and prejudicial, and is due to the prevalent misconception as to the nature of proof. Elsewhere [1] we have shown how continuity may be obtained. It is not difficult to select a group of propositions and to work out with reasonable rigour of logic their interdependence as a single system [2] exhibited to the pupil, at the appropriate stage of his mathematical education (the group in question being chosen for its importance, and of size sufficient to be impressive in the mass) as a model of scientific procedure. This is an experiment that might fruitfully be made in many schools, and its results would be useful when made public. Modern research has shifted the point of emphasis: its aim is now not so much the discovery of already existent and ultimately unchangeable foundations upon which a structure is deductively erected of dependent upon such so-called independent-truths, but rather the creation and exhibition of a group or system of interdependent propositions any one of which may, by suitable transformation of the order of affiliation, itself become regarded and rank as a fundamental element. We venture to commend this view to our fellow teachers as both consonant with the growing spirit of modern philosophy and science and also adapted to free us most rapidly from the burdening shackles of tradition while conserving its essentially permanent values; for it is increasingly clear that all great pioneers, whatever may have been their consciously expressed aims, have subconsciously been ever inspired by the spirit animating the creation of a system of interdependent truths. Thus in quite recent years the concepts of time, space and matter, hitherto conceived for some two thousand years as independent, are now becoming regarded as interdependent.[3] We might

[1] Present work, p. 69; pp. 94 et seq.; pp. 133–6; pp. 139–42; pp. 310–15.

[2] Here may be found a practical solution of a notorious difficulty as to sequence and order in geometrical examinations.

[3] See Dr. A. N. Whitehead, *The Concept of Nature* (Cambridge University Press, 1920)

perhaps venture the further observation, in these dark days of conflict, that such too is the spirit that is arising towards the conciliation of temporal powers—the spirit of *interdependence*, inasmuch as the greatest actions of humanity are the deeds that realize its profoundest dreams or thought. We are passing the stage in which some powers are 'great' and 'independent' with others 'small' and 'dependent', and slowly entering a new orientation of *interdependent* units where none have 'absolute sovereignty'.[1]

We have spoken of qualities lost or diminished: let us speak of the compensating advantages. They are many and great. First and foremost, it is now the exception, not, as too often in previous generations, the rule, to find pupils who take no genuine interest in the study. Far greater numbers now derive educational benefit from geometry. The teacher himself is freer to develop his own particular bent, to profit more fully by his own special experience. A spirit of initiative, of movement, a genuine freshness, blows through the class-room. Save and except with the finest of teachers, such a breeze was rarely to be found there before. Again, geometry is no more an isolated study; it has forged closer and closer bonds with arithmetic, mensuration, algebra, trigonometry, mechanics, and the infinitesimal calculus, giving aid to these and receiving aid from them. Thus has the pupil's mathematical outlook been widened; thus has been provided for and won by his own efforts a more powerful working instrument for practical application. In a word, the subject is alive. We have summed up the position briefly. Let us now consider for a moment the general impressions we have gathered from mathematical teachers on the whole question. We have met few or none who undervalue a preliminary course mainly experimental in substance; but its application in the schools is still too predominantly experimental and neglectful of the function of simple types of reasoning within the capacity of even young children. Such a course is now rightly regarded as indispensable. But the passage from such a predomi-

[1] See the writer's *A New Chapter in the Science of Government* (Chatto & Windus, 1919).

nantly empirical stage to the subsequent conventional forms of treatment is much too abrupt. Continuity should be maintained in greater measure. (See Chapter VIII.)

A small number of experienced teachers exist who prefer, after an age (somewhere between twelve and fourteen) is reached which they consider ripe for the formal logical study of geometry, to return to Euclid (with his definitions, axioms, postulates, and the whole logical apparatus following), provided some reasonable measure of liberty is accorded to the teacher to diverge therefrom in small points where the needs of the class clearly indicate its desirability. Again, a considerable number of teachers would welcome the definite laying-down, by some wide authoritative body, of a fixed order of propositions based on modern geometry and adequately adapted for school use, after the formal study is begun, adding a similar condition to the one just mentioned as regards elasticity of interpretation at the liberty of the teacher, and further, provided such fixed order becomes acceptable for all practical purposes to all educational authorities in the country. But a still larger number appear to think both that the return to Euclid is impracticable, and that no sufficiently wide authoritative body could discover a fixed modern order that all other authorities would accept. Nor, indeed, do they consider a fixed order (whether the old Euclidean or a modern equivalent) desirable in the best interests of a progressive mathematical education. Amongst the reasons—if we interpret them correctly and adequately—influencing this last and weightiest group are these.

In this and other countries long experience, whether national or international, has proved that the attempt to fix a universally recognized order in geometry is impracticable of realization. The whole science of mathematics, and of geometry in particular, is developing its roots—not to speak of its topmost branches—at a rate so rapid that it is reasonably certain that an order fixed and agreed upon to-day would be out of date to-morrow. Further, the natural course of emulation amongst text-book writers and educational publishers may be trusted to evolve in time forms of geometry that are reasonably stable, when mathematical teachers have grown familiar with the changes in the modern basis of

geometric and other mathematical sciences, and have, moreover, received an adequate training in the critique and creations of modern logic and modern psychology. In a word, that it is futile to hope for ideal types of fixed books until the philosophy of the science itself has become settled on a reasonably substantial and stable basis, and the psychology of adolescence also has attained a level more assured and scientific.

A recommendation of the *Conference on the Teaching of Arithmetic* (London County Council, reprinted and provided with an index, 1914), respecting a permanent advisory Board of Studies (pp. 36 and 37, Chap. I), may be referred to here. Such a Board might be useful in connexion with a study of the present question.

We have indicated something of the nature of the many difficulties in the way, either (1) of the establishment of a single and generally acceptable substitute for Euclid, or (2) the rapid emergence of a few reasonably enduring types of substitutes. Of these difficulties two are so far-reaching and so deeply rooted that we ought to scrutinize them somewhat more closely. These we proceed to deal with. First stands[1] the instability of modern philosophy, alike (*a*) general, (*b*) logical, (*c*) mathematical. It is clear to every one whose business it is to follow closely world developments (alike American, Asiatic, and European) and who attempts, as opportunity offers, to contribute his share thereto (however humble the brick he lays in the great palace of truth in each of these branches of thought), that a great system of world-philosophy is becoming gradually created ; and to some extent the broad lines of this great human co-operative design may even now be faintly seen.[2] The majority of mathematical teachers have had neither the leisure, nor the training, to follow these broader developments of their science, even those confined to the particular branch, geometry. They are not, therefore, at present in a position to survey the field effectively, still less to venture themselves on bold voyages of educational experiment.

The second difficulty is only less serious than the first. It is this :

Mathematical teachers in secondary and technical schools seem to be comparatively rare in this country—less so on the Continent—who are familiar even with the elements of modern non-Euclidean Geometry, not to mention other developments of modern mathematical logic. Thus the common pedagogical conception of such central concepts as the straight line, parallel lines, and so on, still move in the narrow, if practical old Euclidean grooves ; and the research now quietly carried on for several generations into the wider significance of the Hellenic geometry is a terra incognita to our schools. Consequently it comes with something of a shock to teachers to discover that these concepts (including the straight line) have become generalized ; and that there are perfectly consistent scientific species of geometry no less valid than the Euclidean (itself, as contrasted with the preceding, being Parabolic Geometry),

[1] See *Janus and Vesta*, Chap. XI (op. cit.).

[2] For an indication of the deep ploughing that is taking place in the ancient field of logic, see *Encyclopaedia of the Philosophical Sciences*: Vol. I: Logic (Macmillan, 1913)—a series of essays by American, French, German, Italian, and Russian thinkers of world-wide reputation. Widely as these essays vary in form from each other, there is a strong subconsciousness of unity inspiring them all, that will be found consonant with the spirit of our own observations.

in which two 'straight lines'[1] in a plane may neither intersect nor be parallel and yet may have a common perpendicular: in which the distance between two 'parallels' diminishes in the direction of parallelism and tends to zero while, in the other direction, the distance increases and tends to infinity: in which the sum of the angles of a triangle may be less than two right angles in which case two (or many[2]) parallels can be drawn through any point to a straight line (all the above in Hyperbolic Geometry): in which as the area of a triangle with real vertices increases, the angles are positive quantities, their sum diminishes and there is a maximum limit to the area of a triangle when the angles all tend to zero (Hyperbolic Geometry): in which the locus of the free extremity of a perpendicular of constant length moving with the other extremity on a fixed straight line is not necessarily parallel to that fixed straight line; and the ideas of parallelism and equidistance may be shown to be quite distinct: in which equidistant straight lines do not exist: (alike in Hyperbolic and in Elliptic Geometry): and in which a straight line may be unbounded yet of finite length (Elliptic Geometry): in which the plane (in Elliptic Geometry) differs from the Euclidean plane (Parabolic Geometry) and from the plane in Hyperbolic Geometry, in such wise that a straight line does not divide it into two distinct regions, for a point may move in the plane from one side of a straight line to the other without crossing it; the elliptic plane is thus a one-sided surface. A simple Moebius sheet illustrates this last quality well. Give a half twist to a long rectangular strip of paper and then gum the ends together; a line traced along the centre of the strip returns to its starting-point but on the opposite surface of the sheet.[3]

Again, the area of a triangle (in Elliptic Geometry) is proportional to the excess of the sum of its angles over two right angles; in Hyperbolic Geometry the area is proportional to the defect.

Though 'Euclidean Geometry is in itself but a degenerate form—in the sense in which a pair of straight lines is a degenerate conic'—yet the needs of common sense and common human experience will always maintain for the Euclidean form of geometry a pre-eminent practical position. It is striking evidence of the penetration and grandeur of the philosophical thought of Euclid and his great predecessors that after the failure of centuries and centuries of effort to deduce it from the remaining postulates modern research has finally justified Euclid's decision in putting the famous fifth statement as to parallels among the postulates and not the axioms, as being an independent assumption indispensable for the genesis of the Euclidean form of geometry as a logical system. But though this is so, not only the general educated public of modern times but the vast majority of mathematical teachers have wrongly concluded that the Euclidean theory of parallels is a necessity of thought, whereas it is but one of many conceivable geometric systems.

We have given these few striking examples of modern

[1] Not of course in precisely the Euclidean sense, inasmuch as the significance of a scientific concept (e. g. straight line) is a function of all the definitions, postulates, and propositions forming that science.

[2] According to the definition of 'parallels'.

[3] See Fig. 97 p. 282 of present work.

discoveries in geometry[1] both to make our own position clear and with the hope of stimulating the interest of young mathematical teachers in the whole subject. We can scarcely with reason expect their elders to face such a revolution in their geometric world of ideas, though we have had the good fortune to find amongst these elders some of the most open-minded, progressive, and initiative spirits. Our own past experience shows us the great difficulty confronting the mind once geometrical ideas have become relatively fixed, even in the early twenties of life, when an effort is made to transcend the orthodox and conventional view of geometry (embracing the structure composed of axioms, postulates, and definitions, partly the result of external sense experience and partly the result of internal intellectual creation), in which one has been trained and educated both at school and university, in the desire to comprehend the wider standpoint of modern discoveries and creations. The difficulty is not unlike what our ancestors must have felt in imagining the antipodes. Instead of reorienting their basal ideas of weight—though the harder in appearance this would have been the easier way in the end—they strove instead to harmonize the concept of a pull always and *absolutely* downwards with a pull always and absolutely *upwards*. Inevitably they found two such self-contradictory conceptions impossible to harmonize, as indeed we should nowadays also: whereas the conception of a pull to the centre of gravity of the earth gradually overcomes the otherwise insuperable difficulty.[2] We are confident that the mathematical teacher would find unexpectedly great profit from a study of Comparative Geometry (for the present this means Euclidean and Non-Euclidean in one system), and find

[1] We have not dealt with the still somewhat inchoate but increasingly promising field of *discontinuous* geometries.

[2] To those interested in a careful discussion, historical and philosophical, of the dominant materialistic philosophy of the nineteenth century, one may favourably recommend a study of Stallo, *Concepts of Modern Physics* (International Scientific Series). Much of it is still applicable to quite recent scientific thought. See also Aliotta, *The Idealistic Reaction against Science* [English translation from the Italian.]

his enjoyment of the study extraordinarily fascinating once the elementary ideas have begun to germinate in his mind.

For the subconscious power of his intellect will, with humility and patience on his part, do the greater half of the work for him ; and few pleasures are keener to the intellect than the observation from time to time of the wonderful growth of new ideas deliberately planted in the garden of the mind, and tended with the same care as a good gardener would give to his flowers. This spontaneous growth, blossoming, and fruiting of new ideas should not be forced (as is too commonly done) to the prejudice of the mental and bodily health ; but needs, we repeat, merely genuine interest, a reasonable measure of periodical attention, humility, and patience. In this kind of reorientation of old mental habits, the student teacher must study 'forwards and backwards', remembering that the initial postulates and the subsequent propositions illuminate each the other, as must be the case with any system of interdependent truths. The teacher must gradually learn to generalize his previous concept of a 'straight line', and in doing so will find himself surprisingly in sympathy with the similar kind of difficulty his pupils find in their respective generalizations of number, quantity, and so forth. The teacher, in fine, will profit greatly by some reasonable acquaintance with the foundations of Comparative Geometry and be in a position to decide with wisdom just what amount of geometry can be best presented to his pupils and with what limitations—and some of the most capable teachers assure us that the pupils profit much by some reasonable understanding of these limitations imposed upon the Hellenic Geometry by modern discoveries. Such a teacher must be prepared to submit himself to the necessary geometric reorientation, to persevere with the study though probably with but a loose grasp for a long time, yet supported by the faith that the hour will at length dawn when the new kingdom of science will burst clearly and enchantingly upon his astonished gaze, and a new mathematical instrument be fashioned to his hand with large increase of freshness and joy, of power and effectiveness even in the teaching of his most elementary pupils.

Real intellectual progress is in general dependent upon such a reorientation of old ideas—ideas which were, in the individual life-history or the history of the race, considered self-evident, and incapable either of limitation or expansion, for long periods. The maintenance of this power of the mind (especially at each crisis of life's several periods) seems to be one of the important conditions of rejuvenation, of revitalization, which unites a wise conservatism of the old and time-tested with movement forwards in the true and the new, and is a quality conducive to a sane and fruitful old age. Conflict of ideas is the price paid for intellectual growth ; and, as a writer in *The Times* once admirably advocated, is perhaps the most potent protection against doctrinairism and its sterner form, fanaticism.

The time may not be far distant when it will be of vital importance for mathematical teachers not merely to recognize that Euclid, though the most practically important, is really only one of many scientific geometries,

but also to have some substantial acquaintance with the science of Comparative Geometry. There are, unfortunately, no particularly easy introductions to modern non-Euclidean geometries yet published; but of fairly elementary books suitable for the secondary school mathematical teacher who is a specialist may be mentioned, in English, Dr. D. M. Y. Sommerville,[1] *Elements of Non-Euclidean Geometry* (G. Bell & Sons, 1914), and another by Dr. H. S. Carslaw (Longmans: Modern Mathematical Series,[2] 1916), each containing a fair historical introduction. Both books may be recommended for English teachers. For those who can read any of the ordinary foreign languages there are excellent text-books, as, for example, in French, German, or Italian (many teachers have returned from the war with a good knowledge of Italian). But, though written expressly for mathematical teachers, these books are, it must be confessed, not easy reading for the average secondary school teacher of mathematics. But once the importance of gaining some familiarity with modern geometry spreads in the schools, doubtless it will not be long before one or two of the more advanced mathematical specialists in the schools will succeed in producing a really easy introduction for the benefit of their less fortunately situated colleagues. One may perhaps venture a hint upon further obstacles which such a book must overcome. In the text-books with which we are acquainted, written for the present purpose, insufficient attention has been given to clearing-up in the early chapters certain difficulties that are not really mathematical but logical and psychological. Moreover, the mathematical authors of such books (otherwise admirable) in general have not enjoyed any wide philosophical training, and have had consequently to rely in this vital sphere of thought upon the opinion of philosophical experts who themselves in turn are insufficiently familiar with mathematical history. It would, therefore,

[1] Dr. Sommerville has also published a most valuable *Bibliography of Non-Euclidean Geometry* (London, Harrison, printed for St. Andrews University Press, 1911).

[2] An admirably stimulating and boldly original series.

seem desirable to preface any attempt of this kind with such a discussion as simple as possible, and specially adapted to facilitate the reorientation of the mind spoken of above. Further, although some history of the subject is generally given in books already published, this part of the explanation might well be made much fuller, starting with a simple account of the geometrical and psychological researches of Proclus [(A.D. 411–85), the great neo-platonic scientist, philosopher, and theologian whose gigantic synthesis of thought has had incalculable influence upon mankind alike in east and in west, and is even yet alive], and carrying the story forwards through many centuries, races, and nationalities to modern times.[1]

Great confusion is now generally rampant in elementary text-books, and even in advanced treatises by eminent mathematicians between the functions of axioms, postulates, and definitions. Thinkers are excessively inclined to be a law each unto himself in these matters.

The following table, to which I am substantially indebted to W. B. Frankland's *Theories of Parallels, an Historical Critique* (Cambridge University Press, 1910), illustrates briefly the confusion into which European[2] editors of Euclid have fallen all down modern centuries in failing to distinguish his postulates from his axioms :

Editor.	*Date.*	*Place.*	*No. of postulates.*	*No. of axioms.*
Ancient.				
Euclid	B.C. 300	Alexandria	5	5
Proclus	A.D. 450	Athens	5	5
Modern.				
Grynaeus	1533	Basle	3	11
Billingsley	1570	London	6	9
Gregory	1703	Oxford	3	12
Playfair	1795	Edinburgh	3	11
Peyrard	1814	Paris	6	9
Todhunter	1862	London and Cambridge	3	12

It is interesting to observe that the above Sir Henry Billingsley was the famous scholar, afterwards Lord Mayor of London, who edited the first printed edition of Euclid's Elements in English ; he was familiar with the Italian Campanus's translation of Euclid from the Arabic into Latin, the Semites (Arabs and Jews) having been the great transmitters of mathematics to Europe from the old Hellenic writings.

[1] Here we may mention the late Dr. W. B. Frankland's various works on Euclid and non-Euclidean geometries.

[2] For the most modern translation see Dr. Heath's *The Thirteen Books of Euclid's Elements* (Cambridge Press), a book which should be in every school library of mathematics.

The axiom (more correctly, a postulate [1]) known as 'Playfair's' was really suggested long before his time either precisely in that form or in a close equivalent. An axiom in Euclid was a common notion, something that the common sense of instructed mankind would accept without proof in any and every field of human inquiry; while a postulate was an assumption of a more specific kind that was laid down for acceptance in a particular science (here geometry), and which, though it might not appear so self-evident as an axiom, even to the instructed intellect, yet could neither be supported by proof nor rejected by disproof.

The whole question of axioms, postulates, definitions, and the logical apparatus of theorems they *engender*, is still in a state of confusion; although considerable advances have been made in clearing up types of old confusion. Of deliberate intent do we use this word *engender* in the above statement. Even by mathematicians of eminence—who have not laboured sufficiently in the field of science that inquires into the mode of genesis of ideas from a psychological and historical standpoint—it is, I venture to think, too commonly thought that once the axioms, postulates, and definitions (or their equivalents [2]) are laid down in a closed system whose units are *necessary*, *sufficient*, and *independent*, or consistent, sufficient, and interdependent, no further basis is necessary for the rigorous deduction of theorems and other logical consequences.

This is not the place to deal in any adequate way with the complex subtleties arising from such a view; but we may point out here that such a system would be quite barren of results without the *engendering or creative* power of the mind which ever is capable of building up new consequences provided it works against a resistance or opposition which is also (as with all natural environments) co-operation—such resistance being given by, and consisting in, the set framework of the system of axioms, postulates, and definitions, deliberately accepted by the mind as delimiting its own creations and thereby giving form and reality to the spiritual will of thinking man. In one sense what we have just said are but truisms; in that no system grows of itself: otherwise the genius and the nincompoop were equal in fruitfulness! But this again leads us to see that the subsequent conclusions are not implicit in, nor embraced by, the initial set or system of propositional forms, but by a process of united deduction (limitation) and induction (creation) come into manifest existence. Thus in every science there is a voluntary element and a necessary element; though a precise delimitation of their spheres and functions would appear to be impossible to man.[3] Without such defining conditions, that is, conditions that produce finiteness, the mind would wander vaguely and create naught but phantoms in an illimitable void.[4]

What conclusion then do we draw from this brief and rough analysis of the genesis of ideas? This. That however rigorously the

[1] In still more recent terminology these older words are becoming replaced by others, upon which it is not necessary here to dwell.

[2] See footnote 1 and p. 82, footnote.

[3] Such also is the living process of judgement in the law courts which both springs from the old law, and yet, transcending it by application to each fresh case, also thereby insensibly creates new law.

[4] See also the writer's *A New Chapter in the Science of Government* (Index, under 'Interdependence').

delimiting systems (axioms, postulates, definitions, propositional forms and so forth) are set forth, and however sharp and rigorous appear the subsequent so-called ‘ deductions ’, the infinite subtlety of man’s intellect is ever discovering, and will assuredly continue to discover, a continuous chain of new and tacit assumptions in the subsequent procedure of so-called ‘deductions ’, for these are the very subconscious creations themselves that engender the objective forms of the science. These assumptions will subsequently be dug out and rank as elements of the original delimiting system ; which system in time will become so complicated as to lead to a necessary reorientation of the whole science ; as indeed is now happening in the kingdom of mathematics after two thousand years of Euclidean kingship. In a word, it is these profoundly deep subconscious assumptions themselves that are the creative additions of man’s thought. In other words, fertile science is the offspring of the engendering intellect of man and the delimiting system of initial conditions consonant with, and yet also re-creating natural phenomena.

As regards definitions, doubtless it was well known to the shrewd Hellenic philosophers (and we note that, in modern times, Peletarius, so long ago as 1557, pointed out), that every axiom that is fertile involves a definition just as every definition that is fertile itself involves an axiom. It is just the above process in the creation and discovery of truth—for the process is both—that forms the difficulty of all logical study and particularly of mathematical study—even of that which has already taken form and shape in the text-book. The mind of the student has not merely to rediscover but to recreate. Hence, too, the sharp limitations to the degree of mastery and advancement of each. So far are the axioms, postulates, definitions, and other primitive principles in any system from being adequate in themselves to produce implicitly their supposed consequences that every mind, from eminent professor to immature schoolboy, must necessarily learn to comprehend in a more or less obscure manner the subsequent chain of reasoning before the axioms, postulates, and definitions themselves become relatively clear. In a word all adequate study, we repeat (in the spirit of Lagrange and Chrystal), has to be backwards and forwards. The student has ever to bear in mind that two things are never equal or equivalent in all respects, without being identically one and the same thing: in which case absolute barrenness obtains.[1] The equation $A=B$ implies some distinction between A and B, wherever a fruitful fact emerges therefrom.

§ VI. *Mechanics.*

On this branch of study I would draw attention to two aspects.

There is first the far-reaching effect which will be surely if slowly produced upon school mathematical education by the modern generalization of the classic Galileo-Newtonian system of mechanics which we may briefly call *Comparative Mechanics*, brought into being by

[1] Mr. G. Goodwill observes on this : ‘ Poincaré sometimes forgets this. Compare the fact that a constant always implies variables.’

a long line of eminent scientists, and particularly indebted to Einstein.[1] This aspect has been briefly and generally considered already. Its ultimate issue can scarcely be less than a wide reorientation of man's view of the mysterious deeps of Nature in its four-dimensionality.

The following quotation from a profound Master of Natural Science[2] is worthy of repeated study:

> 'The materialistic theory (of nature) has a the completeness of the thought of the Middle Ages, which had a complete answer to everything, be it in heaven or in hell or in nature. There is a trimness about it, with its instantaneous present, its vanished past, its non-existent future, and its inert matter. This triteness is very medieval and ill accords with brute fact.
>
> The theory which I am urging admits a greater ultimate mystery and a deeper ignorance. The past and the future meet and mingle in the ill-defined present. The passage of nature, which is only another name for the creative force of existence, has no narrow ledge of definite instantaneous present within which to operate. Its operative presence which is now urging nature forward must be sought for throughout the whole, in the remotest past as well as in the narrowest breadth of any present duration. Perhaps also in the unrealized future. Perhaps also in the future which might be as well as the actual future which will be. It is impossible to meditate on time and the mystery of the creative passage of nature without an overwhelming emotion at the limitations of human intelligence.'

The second aspect I wish briefly to consider is the strengthening of the natural alliance between science and craft.

Many efforts[3] have been made to decide the order in which the various branches of this subject may be best presented to the pupil. These are of real value and contribute much to the understanding of the subject. Within their own proper limits such questions as 'Which should precede—statics or dynamics?' or 'Should they be mingled from the outset?' or 'Which set of units should be introduced and at what stages?' and so forth, are all of great importance, and deserve all

[1] See Bibliography, pp. 379, 380.

[2] See Professor Whitehead, *The Concept of Nature*, p. 73 (Cambridge University Press, 1920).

[3] See particularly the masterly 'Report on the Teaching of Mechanics' published in *The Mathematical Gazette* in issue dated Oct.—Dec. 1918 (G. Bell & Sons, Portugal St., London); syllabuses were contributed by W. J. Dobbs, G. Goodwill, T. P. Nunn, and A. W. Siddons.

the thoughtful consideration that is being devoted to them. But here we aim rather at suggesting broader lines of advance that may contribute, according to the measure of their value, to the development of the new educational conditions that appear to be arising, with certainty if with slowness, from what can be called by no other name than the coming new Era of civilization and culture. Much of what we have put forward will, it is hoped, prove of some use under present conditions that are passing ; much again we venture to hope (if we may judge by the labour in the past it has cost us) will prove of some use in the generation that is coming.

In respect of mechanics our point of view is this. To every science in the world within corresponds a practical craft in the world without. Reciprocally stimulating in this way are theory and practice. Too long has the mechanics of the academic been divorced from the practical activities of the craftsman. In consequence an air of unreality, in spite of the valuable intervention of the experimental laboratory, has wrapped about the teaching of this science of inanimate nature in our schools and colleges.

The practical limitations that arise from such natural phenomena as friction, temperature, and so forth have been unduly ignored in school : the mechanics of the future engineer and academic mechanics are much too foreign to each other. The same complaint is justified in respect of the manual and artistic crafts (carpentry, joinery, masonry, architecture, and so forth). In a word, machine craft and manual craft are both unduly isolated from the academic study of mechanical science. Nor would an impartial observer ever judge, from listening to the average mechanical teaching in school or college, that the human frame itself is a machine of wonderful symmetry, beauty, and power, whose shape and functioning offer the most fascinating illustrations of mechanical science, many of which may be made to appeal to the youngest pupils of this study.

Indeed, it is this very mechanism of the child by which its experiences are gained and realized from its earliest years, thus providing a body of

mechanical knowledge as indispensable and invaluable to the teacher of mechanics as the corresponding geometrical body of experience is to the teacher of geometry. In ultimate analysis the two bodies are, indeed, identical, though for practical teaching purposes they may be usefully treated in separation if the precaution is taken to intertwine them periodically.

We cannot enter here into the many complex questions that demand consideration directly we let our thoughts branch out in this direction of the legitimate spheres and functions of practical craft and theoretical science, and their mutual action and reaction. Elsewhere we have attempted to think out some of its important implications.[1] In respect of mechanics we look forward to the gradual realization of social conditions under which it will be possible to combine, to a far higher degree than has yet been attempted, in one personality genuine love of teaching with an education and training that unite in due measure thorough university scholarship in the science with substantial practical experience in some craft that applies, and is built upon, that science.

Conclusion.

We may generalize the ideal set forth in the last paragraph. Inasmuch as the mathematician in widest sense has three natural fields of application open to him —mechanics, biology, and sociology; and inasmuch as each of these sub-sciences of mathematics draws its life from periodical contact with the corresponding objective crafts, we seem to foresee a time in the perhaps distant future when there will be an adequate number of mathematical teachers combining educational power with high scientific scholarship and a reasonable measure of practical skill in some one of the many fundamental crafts of life to which mathematics is applicable.

[1] See the writer's *Janus and Vesta*, Chap. X; pp. 205, 206; pp. 10–16; Chap. XII; p. 162; p. 285; also the present work, pp. 166–9; and Chaps. XV and XVI.

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